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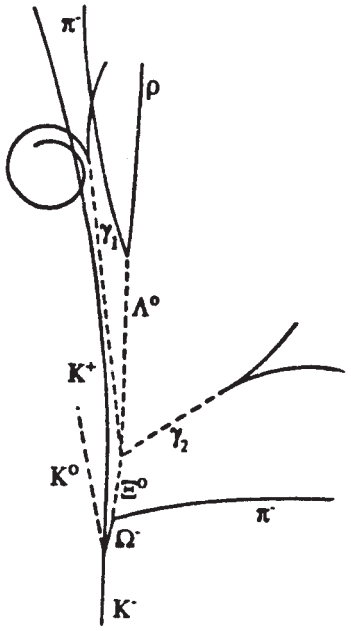

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NEW THERMODYNAMICS: WAVE-PARTICLE DUALITY IN RADIATIVE HEAT TRANSFER

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Abstract

The wave-particle duality of light is accepted. Strangely, when considering the interaction of radiation and matter, the importance of this duality has been overlooked. In part this is due to misunderstandings and their applications in infrared spectrometry. Specifically, in thermodynamics the understanding that photons are particles with momentum has been ignored, while the consideration of EM waves interacting with charge distributions has been embraced. New considerations concerning a photon's momentum in heat transfer will be discussed.

Keywords: Inelastic collisions, wave-particle duality, kinematic heat transfer, radiative heat transfer, infrared spectrometry blunder

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Introduction

Due to the work of the likes of Einstein and Plank, the wave-particle duality of radiation has been fully accepted. Although this duality is a basis for quantum physics, at this time one cannot claim to fully understand light/radiation as both a particle and a wave. Herein, the plausibility of this duality being overlooked in thermal physics will be discussed.

As an EM wave, one often considers light/radiation in terms of Maxwell's equations with interactions existing between electron's charge distributions and the EM waves at various wavelengths/frequencies. What becomes problematic is that heat transfer has certain universal considerations that charge distributions can-not explain.

As a particle, photons can transfer their momentum onto both macroscopic and microscopic objects. For example, on a macroscopic scale, a proposed method of interstellar travel, direct impulse would utilize an energetic beam (microwave?), thus providing impulse upon a sail that powers a craft in outer space [1]. On a microscopic scale, one only needs to consider the photo-electric effect.

If one googles "greenhouse gas experiment," one will find that numerous researchers have shone directional infrared light upon both CO_2 and air, with some interesting results [2-6]. Although not overly scientific, they claim to demonstrate the accepted premise that greenhouse gases are molecules with a lopsided charge distribution, e.g., carbon dioxide (CO_2), water vapour (H_2O), methane (CH_4), etc. Hence, EM waves of specific frequencies can excite the greenhouse gas molecules due to their interactions with the lopsided charge distribution.

The reality is, in such greenhouse gas experiments, both the air and CO_2 heat up, with the heating of the CO_2 always being at a slightly higher rate with a somewhat higher final/limiting temperature. For example, in [6], similar lights with strong infrared component heat a vessel filled with air heated from 23.4 C to 31.6 C, while a similar vessel filled with CO_2 heated from 23.4 C to 36.1 C.

What is problematic in these greenhouse gas experiments is the fact that the air is 99% homonuclear gas [nitrogen (N_2) and oxygen (O_2)]. Both N_2 and O_2 are claimed to be opaque to infrared radiation. Therefore, if accepted, then the minute concentration of CO_2 in air (i.e., 0.04%) is supposedly enough to heat air at a rate that is approximately 65% that of pure CO_2 [based upon (31.6-24.3)/(36.1-24.3)=0.645]. To this author, such an explanation is ridiculous. Accordingly,

although it demonstrates that infrared radiation can heat a gas, the science is lacking.

Allmendinger [7] shone a light source with a strong infrared component into tubes filled with various gases and found temperature increases for all gases tested. Not only does this include homonuclear diatomic gases (e.g., O_2 and N_2) but it also includes monatomic noble gases [Helium (He), Neon (Ne), Argon (Ar)], all of which, according to scientific doctrine, should be opaque to infrared radiation.

Allmendinger [7] further determined that different gases had different experimental limiting temperatures. This would be true if the tube was infinite in length, or the infrared light's intensity was infinite in intensity, because there would be no limiting temperature. This author believes, that given similar intensity and duration, the differences in limiting temperatures would be due to differences in the various gas's infrared absorption cross-sections.

Traditionalists may argue that it was the tube's walls that heated up, without providing any real clarity concerning the mechanism by which the infrared radiation heated the walls. Moreover, if the infrared radiation only heated the tube's walls, then based upon accepted traditional kinetic theory, the expectation would be that the temperature increase would be independent of the gas inside of the tube. This was not the case in Allmendinger's experiments, nor in any of the greenhouse gas experiments. Clearly, different gases are heated by similar infrared radiation at different rates to different limiting temperatures.

Others have also determined interesting results concerning radiative heat transfer [8,9]. In a box, Seim and Olsen [8] hung two folded pieces of aluminum foil, one untouched and the other painted black. The box was then enclosed with clear plastic. First of all, the box was heated by a 500 W tungsten-halogen lamp which can be considered as radiative heat from a blackbody radiation spectrum based upon the Stefan-Boltzmann equation (T=3,200 K) [10]. The black foil's temperature increased faster than the untouched foil. Secondly, the box was heated with a hot plate and the untouched foil heated faster than the black foil.

The black foil adsorbs most of the thermal radiation, which is understandable, since as a blackbody (absorber), the black foil should heat faster when exposed to thermal photons, i.e., the shiny aluminum foil should reflect approximately 90% of the incident thermal radiation [11]. One can then assume that the aluminum foil is primarily heated due to kinematic heat transfer from the surrounding gas molecules, i.e., heat transfer demonstrating the kinetic theory of gases.

Conversely, when heated by the hot plate, the thermal energy transfer is primarily due to intermolecular collisions, i.e., kinematic heat transfer. This author further speculates that in Seim and Olsen, the paint had a rubber component, therefore the surrounding gas molecules bounced off rather than transferred their kinetic energy onto the black foil.

As shown in Fig. 1 and Fig. 2 (sketched from Sein and Olsen [8]) there are significant differences between heat transfer rates from a predominately radiative heat source versus a predominately kinematic heat source. Seim and Olsen [8] unintentionally demonstrated that heat transfer into gases, should be separated into two parts, i.e., radiative heat transfer and kinematic heat transfer. Kinematic heat transfer remains the dominant heat transfer process in many experimental systems, especially those with reflective coatings on their walls, i.e., walls often used in gaseous heat transfer experiments.

Traditionally accepted equipartition-based kinetic theory considers that the gases inside of the box undergo elastic collisions with both the other gases and the system's walls, thus maintaining both their momentum and kinetic energy. This mathematically reinforced conjecture starts with either a gas molecule's momentum or kinetic energy [13] and is generally accepted even though traditionally accepted kinetic theory requires numerous exceptions in order for that theory to match empirical findings.

This author has rewritten kinetic theory [14-16] with the understanding that the system's walls impose their kinematics onto the gas molecules during intermolecular collisions. As illustrated in Fig. 3, the relatively massive wall molecules (1 thru 8), vibrating at temperature, T_w, impose their kinematics upon the enclosed homonuclear gas molecules. This allows for both intermolecular and intramolecular collision to be inelastic [12,14]. Conservation of energy tells us that such inelastic collisions must result in the creation of photons, which can be thought of as thermal photons, i.e., resulting from/in thermal energy exchanges.

The system's wall molecules adsorb and radiate thermal photons, as illustrated in Fig. 3. The absorbed photons become part of the wall molecule's vibrational energy (phonons), which are then recirculated back onto the enclosed system of gas molecules via intermolecular collisions, or thermal radiation. This maintains thermal equilibrium, giving the illusion of elastic collisions [12,14]. It should be emphasized that often experimental apparatus walls are coated with reflective surface (reflect thermal/infrared photons), thus enhancing the illusion. An ensemble

of such inelastic collisions would produce a thermal spectrum, which probably is blackbody in nature.

Consider a homonuclear gas in thermal equilibrium in such an enclosed system. If the walls of the system emit a blackbody radiation, whose peak is centered in infrared, and those walls are in thermal equilibrium with the enclosed gas, then the gas molecules should also adsorb and emit a similar blackbody spectrum, i.e., a blackbody spectrum whose peak is in the infrared [12,14].

The above gives insights into radiative heat transfer versus kinematic heat transfer by a gas and the reason why, in a broad context, heat transfer is related to the interactions of atoms/molecules rather than electron/charge distributions. There is more to this. Statistical thermodynamics defines a system's thermal energy solely in terms of a system's molecule's kinematics. Not only does this make one think in terms of the energetics of the gas molecules, as described in terms of probabilistic elastic collisions, it removes any focus upon the thermal radiation that resides in a gaseous system. This also helps to explain why most scientists/researchers illogically claim that a vacuum has no temperature.

It must be stated that in general, the total energy associated with a gas's kinematics far exceeds the total energy associated with any thermal radiation residing between the gas molecules. Hence when considering the total energy of a gaseous system, the energy associated with the system's radiative component, often can be ignored. Even so, the fact that the radiation travels at the speed of light, while gas molecules move at non-relativistic speeds, means that when considering heat transfer, the radiative component should not be ignored.

The illusion of elastic collisions enhances one's understanding as to why experiments seemingly confirm Boltzmann's mathematical conjecture, which is based upon elastic collisions [12,14]. This reinforces the notions concerning both entropy and the second law based upon experimental evidence. As described by Sheehan, "*The second law of thermodynamics is an empirical law. It has no fully satisfactory theoretical proof. This being the case, its absolute validity depends upon its continued experimental verification in all thermodynamic regimes*" [17]. It must be stated that both entropy and the second law conceptually involve elastic collisions, theories that this author has challenged. [12,14,18-20], along with the problematic general application of elastic collisions to the sciences [21].

Radiative vs Kinematic Heat Transfer

It remains this author's assertion that heat transfer should be divided into radiative heat transfer and kinematic heat transfer [12,14]. Based upon the various experiments [2-9], one must conclude that interactions between the thermal/infrared photons and gases exist. Importantly, this is NOT readily explained in terms of electrons interacting with EM waves, nor is it explainable in terms of traditional statistical thermodynamics.

One must now consider the possibility that in radiative heat transfer, the photon's momentum is passed onto all gases, e.g., both monatomic and polyatomic gases.

Allmendinger's Monatomic Gas Experiment

The case of monatomic gases being heated by infrared radiation is most interesting. Monatomic gases in Allmendinger's experiments [7] heated up with the following findings. The larger the monatomic gas (atom), the higher its witnessed limiting temperature. See Fig. 4 (sketched from Almendinger's [7]). The implication being that the gas molecule/atom has an infrared absorption cross-section that is based upon its physical size. In other words, a larger molecule/atom has a larger infrared absorption cross-section; therefore, it receives a greater impulse over a given duration from a given directional infrared radiation source. Therefore, its limiting temperature would be higher.

Furthermore, Allmendinger determined that monatomic Ar and polyatomic CO_2 experience similar temperature increases and limiting temperatures (see Allmendinger's Fig 25 [7]). This cannot be readily explained in traditionally accepted terms because only carbon dioxide has a lopsided charge distribution that can readily interact with EM waves. That leaves the consideration that they have similar infrared absorption cross-sections, as might be expected based upon carbon dioxide's molecular mass (44.01 grams/mol) being similar to argon's atomic mass (39.96 Da $\approx$ grams/mol) and with the absorption of thermal/infrared radiation now being determined by the incident photon's momentum transfer and the gas molecule's/atom's infrared adsorption cross-sections.

Furthermore, one can speculate that CO_2's lopsided charge distribution was secondary when compared to momentum transfer in radiative heat transfer processes.

It is expected that a gas molecule's size, hence infrared absorption cross-section (cross-sectional area), should be related to its mass. At this point one cannot claim any real knowledge concerning a photon's size, nor how a photon's size (scattering cross-section) can be associated with varying frequencies/wavelengths. Certainly, more experimentation is required to investigate these matters.

Momentum Transfer in a Closed Gaseous System

Consider a closed system of gas, such as that shown in Fig. 3. One could imagine that a polyatomic gas molecule, vibrating at 10^{13} times a second, could receive an impulse from an incident photon striking an atom within the gas molecule, thus stretching the atom's bonds within that gas molecule. In this way the photon's momentum could be transformed into vibrational energy within that gas molecule. Therefore, due to the very high frequency of vibration within a molecule, a mechanism by which vibrational energy is passed onto a gas molecule, surrounded by system walls, can be envisioned.

-Accepting that a closed system's surrounding walls emit blackbody radiation equally in all directions. This means that the gas molecule's translational and/or rotational energy experience a similar net momentum transfer from all directions. Over the duration of a gas molecule travelling from one wall to the next, an equal number of photons will strike a given gas molecule equally from all directions. Therefore, on average, such a gas molecule will receive equal impulses from all directions. Therefore, on average, no net directional momentum would be bestowed upon the enclosed gas molecules by their surrounding blackbody radiation.

In conclusion, on average, a polyatomic gas molecule, in a closed system containing uniform radiation, could attain a net vibrational energy, but not necessarily a net translational nor rotational energy from their surrounding infrared photon's momentum. Similarly, a monatomic gas molecule in such a closed system, would, on average, attain no net energy from its surrounding radiation.

Next consider blackbody radiation being the result of inelastic molecular collisions. As previously discussed by this author [12,14], not only is the blackbody radiation emitted by the walls due to an ensemble of inelastic molecular collisions, but such thermal/blackbody radiation would also be emitted by an ensemble of polyatomic gas molecules. This would be due to inelastic atomic collisions within the gas molecules.

Accordingly, when in thermal equilibrium, the gas molecules must absorb as much thermal energy from incident photons, as they emit as blackbody radiation. This does not necessarily mean that they absorb and emit photons at identical wavelengths/frequencies. Rather, the total thermal energy emitted must equal the total thermal energy absorbed, when in thermal equilibrium

Infrared Spectrometry Blunder

One must ask why the grandiose mistake by the sciences? Arguably, the oversight concerning a photon's momentum in heat transfer is a result of numerous misunderstandings. A significant one being what this author considers to be the infrared spectrometry blunder. It starts by the acceptance that the walls of an infrared spectrometer radiate a blackbody spectrum whose peak is in the infrared with radiation being uniform in all directions, when the system is in thermal equilibrium.

Consider how one operates an infrared spectrometer. One first of all evacuates the spectrometer and then measures the background blackbody radiation sometimes referred to as running a blank. When a polyatomic gas is inserted into the spectrometer, the ensemble of gas molecules absorbs and radiates a similar blackbody spectrum. When in thermal equilibrium that gas's thermal signature must be the same spectrum that one measured and then subtracted when running the blank. In other words, the practice of subtracting the background blackbody spectrum in infrared spectrometry, is the art of subtracting a gas' thermal signature.

What then is the output spectrum currently provided by an infrared spectrometer? It is a spectrum that is based solely upon the interactions of gas molecules with the EM waves associated with the radiation. Certainly, such a spectrum can be unique to the specific gas being tested. Hence it can identify the gas, however, it should not be used to determine all of the gas' thermal behaviours.

No wonder the traditionally accepted sciences were so emboldened as to ignore a photon's momentum, even though it is the photon's momentum that adds to matter's vibrational energies that being energy associated with heat.

Clearly, the sciences have focused upon a photon's electromagnetic (EM) wave interactions with matter. In so doing, scientists remain oblivious to a photon's duality, even though one embraces phenomena that take into account a photon's momentum, i.e., the photoelectric effect.

Homonuclear Gases

No wonder the claim exists that homonuclear gases are opaque to infrared frequencies. The reality is that such gases do not absorb energy associated with EM waves because their charge distribution is not lopsided. However, they certainly can absorb an infrared photon's momentum, as can all gases, as was shown by Allmendinger [7].

The implications to the modeling in global warming is of planetary importance, as has been previously discussed by this author [14,22].

All Matter

Blackbody radiation emanates equally from all surrounding system's walls (e.g., a traditional experimental system). Now consider a directional stream of photons impacting any gas. That gas will witness a net momentum increase. Hence a kinetic energy increase, as demonstrated by their temperature increase. One can now envision a gas attaining a net translational and/or rotational energy due to a directional stream of photons, as proven by the previously discussed experiments [2-9].

One can further extend this to most all matter. Specifically, photon's can pass their momentum onto most matter. This may result in a momentum increase of that matter, or a temperature increase within that matter. An interesting question at this point, left onto others becomes: How exactly does this vary with frequency?

Conclusions

The particle-wave duality of photons has not been properly addressed in thermodynamics. It is as if the duality is limited to quantum physics. In order to fully understand how all matter interacts with thermal radiation, one must consider all aspects of a photon's duality. In so doing, one attains a more universal understanding.

There are numerous experiments where directional light with a strong infrared component is shone upon gaseous systems. In such experiments different gases have been determined to absorb different amounts of radiation, by their differing elevated temperatures. Those vested in traditional thermodynamics may argue that the radiation is heating the walls and not the gas. However, this would mean that the experimental limiting temperature would be independent of the gas. This is

clearly not the case, as different gases possess different limiting temperatures. Therefore, there must be a gas dependence.

This author has speculated that the dependence is based upon the gas molecule's size. Moreover, the implication becomes that directional radiation passes a directed impulse onto the gas molecules. In which case, a larger gas molecule will possess a larger infrared absorption cross-section, hence will heat up to a higher experimental dependent limiting temperature. This was clearly witnessed by various experimenters, whose work has largely been ignored due to its challenging accepted doctrine.

The notion that a system's total thermal energy is limited to the kinematics of the matter within that system is one that ignores the fact that radiation is energy within a system and can be thermal in nature. Understandably, a system's total energy can generally be approximated by the energy associated with its matter's kinematics. However, due to the speed of light, the energy associated with thermal radiation cannot always be ignored, especially when considering heat transfer.

The misunderstandings have origins in the emphasis that is put on statistical thermodynamics. As originally envisioned by Boltzmann, Maxwell and others, this is the practice of wrongly placing complete belief in elastic collisions on a molecular scale. Reality is that such collisions are inelastic. Furthermore, experimental systems are plagued by the illusion of elastic collisions, where the energy from thermal photons resulting from inelastic collisions are recirculated back into that system.

Such traditionally misplaced logic has been reinforced by the infrared spectrometry blunder. This has led to the misguided assertion that certain gases do not absorb infrared frequencies of radiation. Reality is that all gas molecules can absorb the momentum associated with photons as particles, and can be associated with blackbody radiation. Lopsided charge distributions in gases do enable those gases to interact with the electromagnetic nature of photons, which is not necessarily the gas' true thermal signature.

Figures

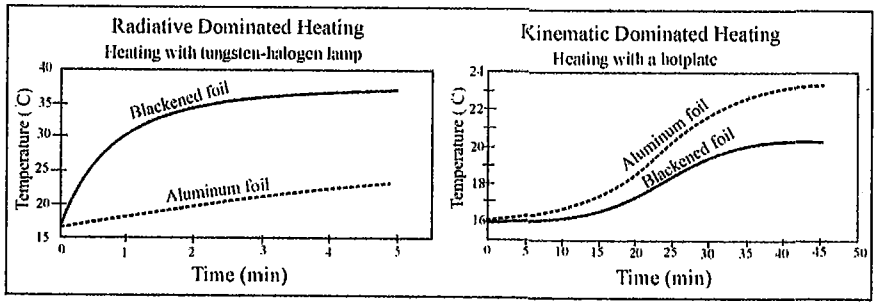

Fig. 1 on the left: Sketch of radiative dominated heat transfer, as determined in the experiment by Seim and Olsen [8]. (Sketched from Fig. 2 of Seim and Olsen [8])

Fig. 2 on the right: Shows kinematic dominated heat transfer, as determined in the experiment by Seim and Olsen [8]. (Sketched from Fig. 3 of Seim and Olsen [8])

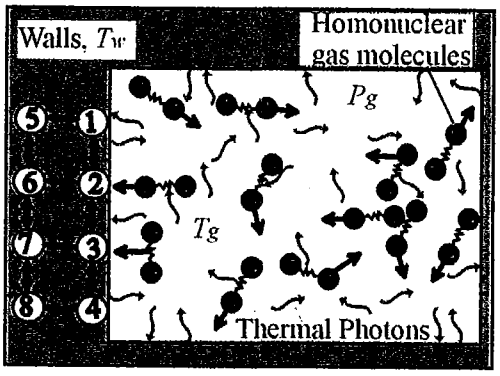

Fig. 3: Shows a homonuclear gas at temperature (T_g), intermixed with thermal photons in a fully enclosed system.

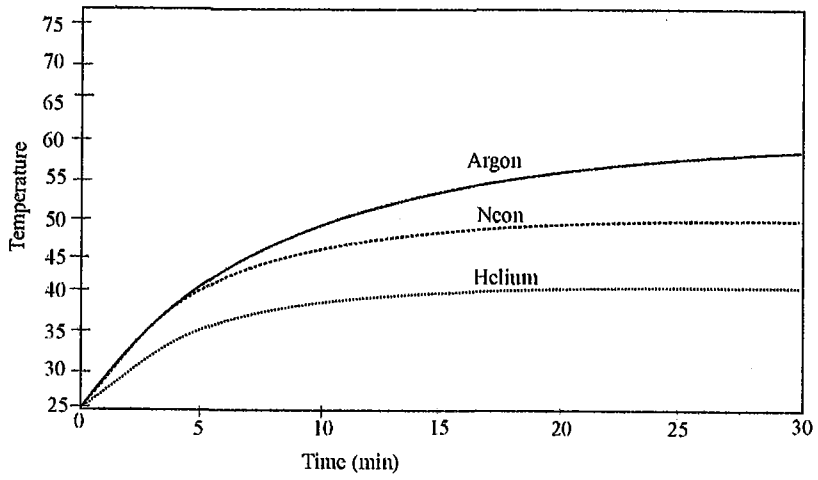

Fig. 4: Sketch showing the temperature increase and limiting temperature, as determined in the experiment on monatomic gases by Allmendinger [7]. (Sketched from Fig. 26 of Allmendinger [7])

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THE HELIX CURVES ON THE SPHERES OF LORENTZ-MINKOWSKI SPACE R^3_1

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Abstract

Aydinalp and Ugurlu [1] defined the causal spirals on the spheres of the Lorentz-Minkowski 3−space $\mathbb{R}^3_1$. In this paper, we obtain differential equations of helix curves on the spheres of the space $\mathbb{R}^3_1$ and determine the helix curves which are solutions of these equations.

keywords: Lorentz-Minkowski 3−Space, Lorentzian spherical helices, Hyperbolic helix, Light-Conical helix.

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1 Introduction

Helices are special curves that are widely used in differential geometry of space curves. A helix in 3−dimensional Euclidean space E^3 is defined by the property that its tangents makes a constant angle with a fixed line. A classical result stated by M. A. Lancret(1902) and proved by B. de Saint Venant (1845) [2]: "A necessary and sufficient condition in order to a curve be a general helix is that the ratio of curvature to torsion be constant.". This specificity of a helix takes us to the definition of different helices.

In [3], some characterizations for a generalized helix are given by using harmonic curvatures of a curve. Scofield obtained the integral solution of natural equations of curves of constant slope or general helices and gave explicit parameterization for spherical helices [4].

The concept of slant helix has introduced by Izumiya and Takeuchi [5]. This type of helix is given by the property that principal normal lines make a constant angle with a fixed direction. For a resent survey on the helices and spherical curves of the Euclidean 3-space E^3, see [6], [7], [8], [9], [10].

In analogous way in E^3, Ali and Lopez [11] studied the causal helices in Minskowski space R_1^3. All helices in the space R_1^3 are completely classified by Walrave in [12].

The helices play important roles in the study of the geometry of the curves. Also those curves can be applied to science and engineering, such as examining the structure of the DNA molecule, the space motions, the motion of particles in a magnetic field and the structure engineering [13]. Generally, some other kind of helices and their results are studied by many mathematicians [14],[15], [16], [5].

Barros et al. proved that general helices in the Lorentzian space forms correspond with non-null curves in L^3 for which the ratio of curvature to torsion is constant. Furthermore they show that general helices in 3-dimensional Lorentz-Minkowski space correspond with geodesics either of right general cylinders or flat B-scrolls [17].

In [18], Ferrandez et al. express and prove a Lancret-type theorem for null generalized helices in Lorentz-Minkowski space R_1^n and express that the only null generalized helices are the ordinary null helices.

Inoguchi and Lee [19] examine null curves in 3-dimensional Minkowski

space. They give some characterizations of null helices in terms of their associated curves and obtain new results on null helices. Furthermore, the other characterizations of null helices in the space R_1^3 are given by in [20], [21], [14], [22], [23].

Ali et al. define the new types of the helices in the space-time R_1^4 called $k-$type slant helices, for $0 \leq k \leq 3$. They gave some characterizations dealing with these helices [24].

As an another null helix, the null generalized slant helix in the space R_1^3 is defined by Karadag's [16]. Authors give some characterizations for these special helices. Furthermore, the null helices, null slant helices and Cartan slant helices are examined by using the notions of the associated curves by Coi and Kim [25]. Authors give some properties of spacelike curve with a null principle normal vector field and important characterizations.

In [17], authors obtained the representations formula of pseudo null curves by means of structure functions and gave some characterizations of the $k-$type pseudo lightlike helices.

Slant helices in the anti-De-sitter space H_1^3 are defined by Lucas and Yagues in [26]. The authors gave the characterizations according to the curve's curvature and a method to construct all the slant helices in H_1^3.

Ugurlu et al. [27], [28] obtained the differential equations of timelike helices on the surface of the Lorentzian sphere with radius r and gave the parametrizations of these curves. Furthermore, they found the invariants and the instantaneous rotation vectors of the Frenet and Darboux frames.

For a resent survey on the helices of the Lorentz-Minkowski 3-spaces, see [29], [30], [31], [32], [33], [34], [35], [23], [36], [37], [38].

In this study, we obtain differential equations of all helices on the hyperbolic, Lorentzian and lightlike (null) spheres of the Lorentz-Minkowski space R_1^3 by using Lorentzian angles and find general parametrizations of these curves.

2 Basic Concepts

Let $\mathbb{R}_1^3$ be a Minkowski 3-space provided with natural Lorentzian metric given by

$$\langle .,. \rangle = dx_1^2 + dx_2^2 - dx_3^2, \tag{1}$$

where (x_1, x_2, x_3) is a standard rectangular coordinate system of $\mathbb{R}_1^3$. An arbitrary vector $\boldsymbol{v} = (v_1, v_2, v_3)$ in $\mathbb{R}_1^3$ is said to be *spacelike (resp. timelike, lightlike)* if $\langle \boldsymbol{v}, \boldsymbol{v} \rangle > 0$ or $\boldsymbol{v} = 0$ (resp. $\langle \boldsymbol{v}, \boldsymbol{v} \rangle < 0$, $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0$ and $\boldsymbol{v} \neq 0$). The *norm* of a vector $\boldsymbol{v} \in \mathbb{R}_1^3$ is defined by $\|\boldsymbol{v}\| = \sqrt{|\langle \boldsymbol{v}, \boldsymbol{v} \rangle|}$. Thus, a spacelike (timelike) vector $\boldsymbol{v}$ is *unit* if $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 1$ ($\langle \boldsymbol{v}, \boldsymbol{v} \rangle = -1$).

The spheres of the space $\mathbb{R}_1^3$ are defined as follows:

The set of all spacelike vectors of radius $r > 0$ with origin-centered is called *Lorentzian sphere* of radius r and denoted by

$$S_1^2(r) = \left\{ \boldsymbol{v} \in \mathbb{R}_1^3 \,|\, \langle \boldsymbol{v}, \boldsymbol{v} \rangle = r^2 \right\}.$$

The sphere $S_1^2(r)$ is a Lorentzian 2-manifold of constant sectional curvature $1/r^2$. For $r > 0$, the quadric

$$H_0^2(r) = \left\{ \boldsymbol{v} \in \mathbb{R}_1^3 \,|\, \langle \boldsymbol{v}, \boldsymbol{v} \rangle = -r^2 \right\}$$

is called *hyperbolic sphere* of radius r. The sphere $H_0^2(r)$ is also a Riemannian 2-manifold of constant sectional curvature $-1/r^2$. This quadric has two connected components given by

$$H^+(r) = \left\{ \boldsymbol{v} \in H_0^2(r) \,|\, \langle \boldsymbol{v}, \boldsymbol{e_3} \rangle < 0 \right\},$$

$$H^-(r) = \left\{ \boldsymbol{v} \in H_0^2(r) \,|\, \langle \boldsymbol{v}, \boldsymbol{e_3} \rangle > 0 \right\}.$$

The surface $H_0^2(r)$ is the hyperboloid model of hyperbolic geometry from non-Euclidean geometry in the space $\mathbb{R}_1^3$. The set Λ^2

$$\Lambda^2 = \left\{ \boldsymbol{v} \in \mathbb{R}_1^3 - \{0\} \,|\, \langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0 \right\}$$

of all lightlike vectors with length $r = 0$ is called *lightlike cone* of $\mathbb{R}_1^3$. This quadric has also two components given by

$$\begin{aligned} \Lambda^+ &= \left\{ \boldsymbol{v} \in \Lambda^2 \,|\, \langle \boldsymbol{v}, \boldsymbol{e_3} \rangle < 0 \right\}, \\ \Lambda^- &= \left\{ \boldsymbol{v} \in \Lambda^2 \,|\, \langle \boldsymbol{v}, \boldsymbol{e_3} \rangle > 0 \right\}. \end{aligned} \tag{2}$$

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two vectors in the space $\mathbb{R}^3_1$. The *Lorentzian cross product* of $\boldsymbol{u}$ and $\boldsymbol{v}$ is given by

$$\boldsymbol{u} \times \boldsymbol{v} = (u_3v_2 - u_2v_3,\ u_1v_3 - u_3v_1,\ u_1v_2 - u_2v_1)\,, \tag{3}$$

where $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$. For standard base vectors, we have

$$\boldsymbol{e_1} \times \boldsymbol{e_2} = \boldsymbol{e_3},\quad \boldsymbol{e_2} \times \boldsymbol{e_3} = -\boldsymbol{e_1},\quad \boldsymbol{e_3} \times \boldsymbol{e_1} = -\boldsymbol{e_2}\ .$$

Let V be a 2-dimensional linear subspace $\mathbb{R}^3_1$. Then, there are three mutually exclutive possibilities for V:

(i) V is called to be *spacelike* if the restriction $\langle .,. \rangle\,|_V$ of the Lorentzian metric on V is positive definite.

(ii) V is called to be *timelike* if the restriction $\langle .,. \rangle\,|_V$ of the Lorentzian metric on V is Lorentzian, i.e, non-degenerate and of the signature (1). Then, V is a timelike plane.

(iii) V is *lightlike* (or *null*) if $\langle .,. \rangle\,|_V$ is degenerate.

The components Λ^+ and Λ^- is called *future cone* and *past cone*, respectively. Then a ray in Λ^+ stating at the origin corresponds to a point on boundary of H^3. The set of such rays form the sphere at infinity $S^2_\infty = \partial H^3$.

For an arbitrary point $\boldsymbol{u}$ in Λ^+ the *horosphere* is defined by

$$h_u := \left\{ \boldsymbol{x} \in H^+ \,|\, \langle \boldsymbol{x}, \boldsymbol{u} \rangle = -1 \right\},$$

which inherits an Euclidean structure.

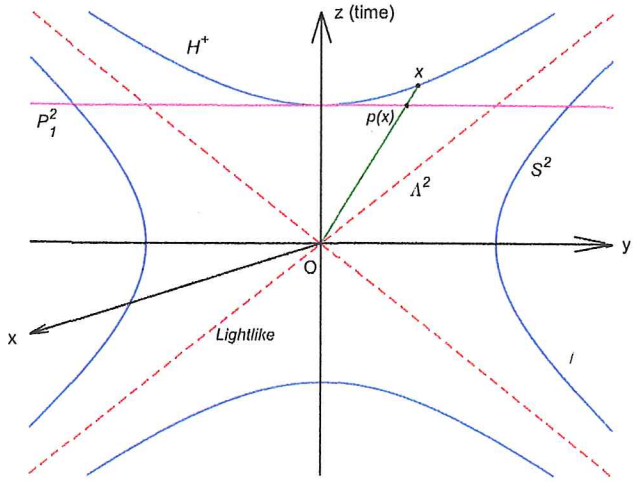


Figure 1: The quadrics of the 3−dimensional Lorentzian space [33].

Let denote the radial projection by p. Then the projection is given by

$$p : \{\boldsymbol{x} \in \mathbb{R}_1^3 \,|x_3 \neq 0\} \to P_1^2 = \{\boldsymbol{x} \in \mathbb{R}_1^3 \,|x_3 = 1\},$$

where P_1^2 is affine plane along the rays through the origin. The projection p is a homeomorphism from H^+ onto the 2−dimensional open unit ball B^2 in P_1^2 centered at the origin $(0,0,1)$ of P_1^2, which give the projective model of $\hat{H}^3$. The affine plane P_1^2 contains B^2 and its set theoretic boundary ∂B^2 in P_1^2, which is identified with S_∞^2. Then we have $\bar{B}^2 = B^2 \cup \partial B^2$.

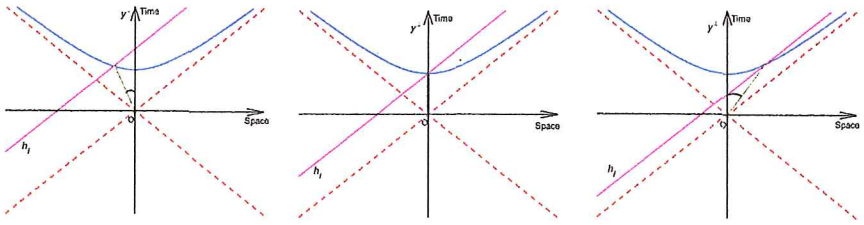
Now we define the *geodesic plane* $\boldsymbol{u}^\perp$ for an arbitrary point $\boldsymbol{u}$ in S_1^2 as

$$\boldsymbol{u}^\perp := \{\boldsymbol{x} \in H^3 \,|\langle \boldsymbol{x}, \boldsymbol{u}\rangle = 0\}.$$

A dual point $\boldsymbol{u}$ also defines a *half-space* in H^3 given by

$$\Pi_u := \{\boldsymbol{x} \in H^3 \,|\langle \boldsymbol{x}, \boldsymbol{u}\rangle \leq 0\}.$$

Definition 1 *The signed distance d between horosphere and a plane (respectively, point, horosphere) is the distance by which the horosphere extends past the plane (respectively, point, horosphere). The distance d may be positive, negative, or zero, as shown in (Fig. 2). (For the concepts defined in this section, see [38]).*



(a) $d > 0$ (b) $d = 0$ (c) $d < 0$

Figure 2: The signed distance from a plane to a horosphere is the distance d by which the horosphere extends to the plane.

3 The Lorentzian Angles in The Minkowski Space $\mathbb{R}^3_1$

Let $\boldsymbol{u}, \boldsymbol{v}$ be two vectors in the space $\mathbb{R}^3_1$. The Lorentzian angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined with respect to the causal characters of the vectors as follows [39], [36], [23], [40], [41]:

(i) Let $\boldsymbol{u}, \boldsymbol{v}$ be two timelike vectors. Then $|\langle \boldsymbol{u}, \boldsymbol{v} \rangle| \geq \|\boldsymbol{u}\| \, \|\boldsymbol{v}\|$ and equality holds if and only if $\boldsymbol{u}$ and $\boldsymbol{v}$ are proportional. In this case that both vectors lie in the same cone, there exists a unique real number $\theta \geq 0$ such that

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = - \|\boldsymbol{u}\| \, \|\boldsymbol{v}\| \cosh \theta. \tag{4}$$

This number is called the *hyperbolic angle* between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

(ii) Then the number $\theta \geq 0$ that satisfies

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \|\boldsymbol{u}\| \, \|\boldsymbol{v}\| \cosh \theta. \tag{5}$$

is called the *central angle* between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

(iii) Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be spacelike vectors in $\mathbb{R}^3_1$ and they span a spacelike vector subspace. Then $|\langle \boldsymbol{u}, \boldsymbol{v} \rangle| \leq \|\boldsymbol{u}\| \, \|\boldsymbol{v}\|$ and equality holds if and only if $\boldsymbol{u}$ and $\boldsymbol{v}$ are proportional. In this case, the number $\theta \geq 0$ that satisfies

$$|\langle \boldsymbol{u}, \boldsymbol{v} \rangle| = \|\boldsymbol{u}\| \, \|\boldsymbol{v}\| \cos \theta. \tag{6}$$

is called the *spacelike angle* between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

(iv) Let $\boldsymbol{u}$ be a spacelike vector and $\boldsymbol{v}$ be a timelike vectors in $\mathbb{R}_1^3$. Then there is a unique real number $\theta \geq 0$ such that

$$|\langle \boldsymbol{u}, \boldsymbol{v} \rangle| = \|\boldsymbol{u}\| \, \|\boldsymbol{v}\| \sinh \theta. \tag{7}$$

This number is called the *timelike angle* between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

(v) Two dual geodesic planes $\boldsymbol{u}^{\perp}$ and $\boldsymbol{v}^{\perp}$ do not intersect in H^2 but intersect in ∂H^2 if and only if

$$\left| \left\langle \tilde{a}, \tilde{b} \right\rangle \right| = \cos \bar{0} = 1. \tag{8}$$

In this case the *Lorentzian angle* between them is zero number 0. It must be note that the Lorentzian angles correspond to hyperbolic distances between two points on the hyperboloid model $H^+(r)$.

(vi) Let $\boldsymbol{u} \in \Lambda^+$ and $\boldsymbol{v} \in S_1^2$. The *lightlike angle* between them is signed distance d between $\boldsymbol{v}$ and h_u. This angle gives the equality as

$$|\langle \boldsymbol{u}, \boldsymbol{v} \rangle| = e^{-d}. \tag{9}$$

(vii) Let $\boldsymbol{u} \in \Lambda^+$ and $\boldsymbol{v} \in H^+$. The *lightlike angle* between them is signed distance d between $\boldsymbol{v}^{\perp}$ and h_u. This angle gives the equality as

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = -e^{-d}. \tag{10}$$

(viii) Let $\boldsymbol{u}, \boldsymbol{v} \in \Lambda^+$.The *lightlike angle* between them is signed distance d between h_u and h_v. This angle gives the equality as

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = -2e^{-d}. \tag{11}$$

(For the geometrical interpretations of Lorentzian angles, see [40], [38]).

Definition 2 *Let $\boldsymbol{\alpha}(t)$ be a differentiable spherical curve at the space $\mathbb{R}_1^3$. The curve $\boldsymbol{\alpha}(t)$ is called a hyperbolic (resp. Lorentzian, conical) helix if all tangent vectors of $\boldsymbol{\alpha}(t)$ make a constant timelike (resp. (hyperbolic, timelike and null), timelike) angle with the $z-$time axis.*

4 The Helices on The Hyperboloid Model $H^+(r)$

The hyperbolic sphere of center O and radius r is the surface given by

$$H_0^2(r) = \left\{(x,y,z) \in \mathbb{R}_1^3 \left| x^2+y^2-z^2 = -r^2 \right.\right\}.$$

We observe that the set $H_0^2(r)$ has exactly two connected components. Thus, the hyperboloid model, the positive sheet of H_0^2, is denoted by

$$H^+(r) = \left\{(x,y,z) \in \mathbb{R}_1^3 \left| x^2+y^2-z^2 = -r^2, z>0 \right.\right\}.$$

This is a spacelike surface and called a hyperbolic plane [38], [40], [41]. We shall compute the first fundamental form of $H^+(r)$ at a point of the coordinate neighborhood given by parametrization

$$\boldsymbol{X}(u,v) = (r\sinh u\cos v, r\sinh u\sin v, r\cosh u), \tag{12}$$

where $0 \leq v \leq 2\pi$ and $u \in \mathbb{R}$.

First, observe that

$$\left.\begin{array}{ll} \boldsymbol{X}_u(u,v) = & (r\cosh u\cos v, r\cosh u\sin v, r\sinh u) \\ \boldsymbol{X}_v(u,v) = & (-r\sinh u\sin v, r\sinh u\cos v, 0). \end{array}\right\} \tag{13}$$

Hence,

$$\left.\begin{array}{lll} E(u,v) = & \langle \boldsymbol{X}_u, \boldsymbol{X}_u\rangle = & 1 \\ F(u,v) = & \langle \boldsymbol{X}_u, \boldsymbol{X}_v\rangle = & 0 \\ G(u,v) = & \langle \boldsymbol{X}_v, \boldsymbol{X}_v\rangle = & r^2\sinh^2\theta. \end{array}\right\} \tag{14}$$

Thus, if w is a tangent vector to the sphere at the point $\boldsymbol{X}(u,v)$, given in the basis associated to $\boldsymbol{X}(u,v)$ by

$$\boldsymbol{w} = a\boldsymbol{X}_u + b\boldsymbol{X}_v$$

then the square of the length of w is given by

$$|\boldsymbol{w}|^2 = I(w) = Ea^2 + 2Fab + Gb^2 = r^2a^2 + b^2r^2\sinh^2\theta.$$

Let us determine the curves in this coordinate neighborhood of the hyperboloid model $H^+(r)$, which make a constant angle θ with the $z-$axis.

Theorem 3 *Let θ be constant timelike angle between the $z-$ time axis and the spacelike curve $\boldsymbol{\alpha}_{\boldsymbol{\lambda}}(t)$, and assume that $\lambda = \operatorname{csch}\theta$. The parametrization of all helices on the surface of the hyperboloid model $H^{+}(r)$ is as follows:y*

$$\boldsymbol{\alpha}_{\boldsymbol{\lambda}}(t) = (\cosh t \cos v(t), \cosh t \sin v(t), \sinh t), \tag{15}$$

where

$$v(t) = \frac{\sqrt{2}\cosh t\sqrt{\lambda^2+\operatorname{sech}^2 t}}{\lambda^2+\lambda^2\cosh(2t)+2}\left[\lambda\sqrt{\lambda^2+1}\sinh^{-1}\left(\frac{1}{\sqrt{\lambda^2+1}}\sinh t\right)\sqrt{\frac{\lambda^2+\lambda^2\cosh(2t)+2}{\lambda^2+1}}\right.$$

$$\left.+\sqrt{\lambda^2+\lambda^2\cosh(2t)+2}.\tan^{-1}\left(\frac{\sqrt{2}\sinh t}{\sqrt{\lambda^2+\lambda^2\cosh(2t)+2}}\right)\right]+c, \quad \lambda = \operatorname{csch}\theta \tag{16}$$

in which c is constant of integration.

Proof. We may assume that the required curve

$$\boldsymbol{\alpha}(t) = \boldsymbol{X}(u(t), v(t)) = (r\sinh u(t)\cos(v,t), r\sinh u(t)\sin(v,t), r\cosh u(t))$$

is the image $\boldsymbol{X}$ of the curve $(u(t), v(t))$ of the $uv-$plane with $v = v(\lambda, t)$. In this case we have

$$\begin{aligned}\boldsymbol{\alpha}'(t) = (&ru'\cosh u\cos v - rv'\sinh u\sin v,\\ &ru'\cosh u\sin v + rv'\sinh u\cos v,\ ru'\sinh u).\end{aligned} \tag{17}$$

Thus, the constant timelike angle between the tangent of the curve and the $z-$axis is real number θ satisfying the equality

$$|\langle \boldsymbol{e_3}, \boldsymbol{\alpha}'(t)\rangle| = |\boldsymbol{\alpha}'(t)|\sinh\theta$$

or

$$|-u'\sinh u| = \sqrt{v'^2\sinh^2 u + u'^2}\sinh\theta, \tag{18}$$

since in the basis $\{\boldsymbol{X}_u, \boldsymbol{X}_v\}$ the vector $\boldsymbol{\alpha}'(t)$ has coordinates (u', v') and the vector $\boldsymbol{X}_u$ has coordinates $(1, 0)$. By (18) we obtain the differential equation

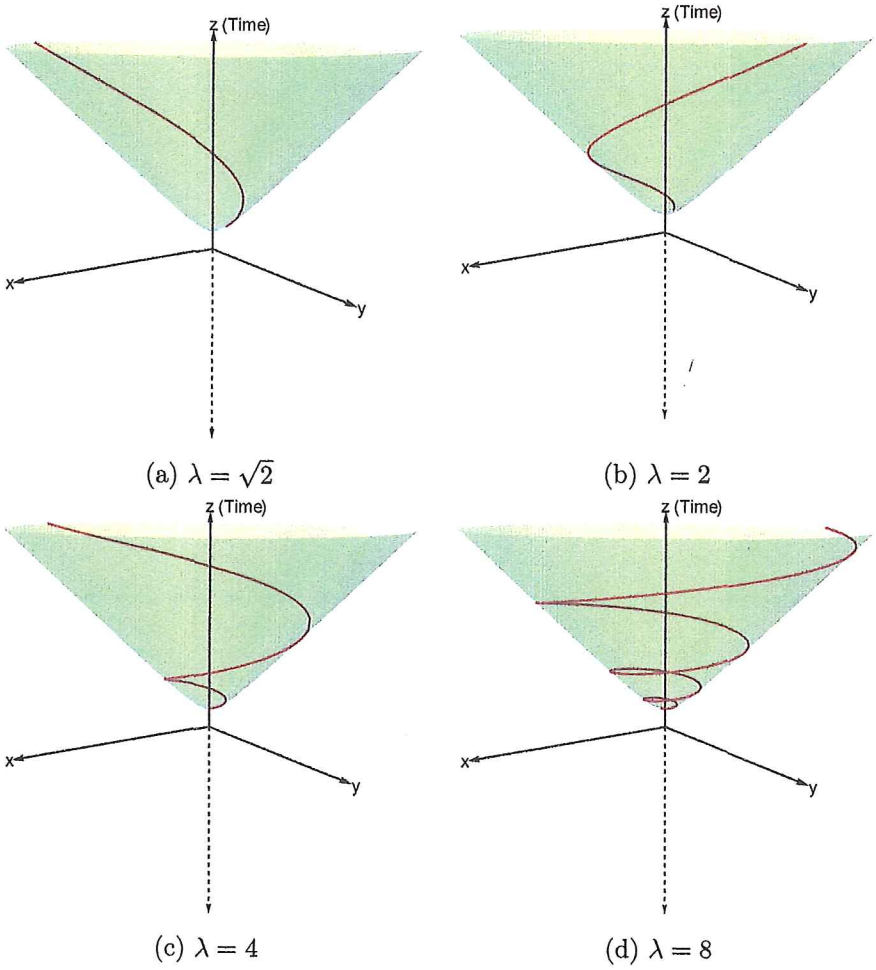
$$\left(\frac{dv}{du}\right)^2 \sinh^2 u + 1 = \frac{\sinh^2 u}{\sinh^2 \theta} \tag{19}$$

or

$$\frac{dv}{du} = \sqrt{\frac{1}{\sinh^2 \theta} - \frac{1}{\sinh^2 u}}. \tag{20}$$

Hence, by integration, we obtain $v(t)$ given by (16). ■

Considering (16) and $\lambda = \operatorname{csch} \theta$ the parametrization of the helices making the constant timelike angle with $z-$time axis is given by (15).



(a) $\lambda = \sqrt{2}$ (b) $\lambda = 2$

(c) $\lambda = 4$ (d) $\lambda = 8$

Figure 3: Helices on the hyperbolic model $H^+(r)$ for different values of λ

5 Helices on The Lorentzian Sphere $S_1^2(r)$

Since the Lorentzian sphere $S_1^2(r)$ is a timelike surface, we know that the helices on this surface are divided into three classes as timelike, spacelike and lightlike helices. The lightlike helices are the straight lightlike lines. In this section we find timelike and spacelike helices on the surface of the

Lorentzian sphere $S_1^2(r)$.

Lets consider the parametrization given by

$$\boldsymbol{X}(u,v) = (r\cosh u\cos v, r\cosh u\sin v, r\sinh u), \tag{21}$$

of the sphere $S_1^2(r)$ with $0 \leq v \leq 2\pi$ and $u \in \mathbb{R}$. We may assume that the required curve

$$\begin{aligned}\boldsymbol{\beta}(t) &= \boldsymbol{X}(u(t), v(t)) \\ &= (r\cosh u(t)\cos v(t), r\cosh u(t)\sin v(t), r\sinh u(t))\end{aligned} \tag{22}$$

is the image by $\boldsymbol{X}$ of a curve $(u(t), v(t))$ of the uvplane.

First, observe that

$$\left.\begin{aligned}\boldsymbol{X}_u(u,v) &= (r\sinh u\cos v, r\sinh u\sin v, r\cosh u) \\ \boldsymbol{X}_v(u,v) &= (-r\cosh u\sin v, r\cosh u\cos v, 0).\end{aligned}\right\} \tag{23}$$

Hence, the coefficients of the first fundamental form are

$$\left.\begin{aligned}E(u,v) &= \langle \boldsymbol{X}_u, \boldsymbol{X}_u\rangle = -r^2 \\ F(u,v) &= \langle \boldsymbol{X}_v, \boldsymbol{X}_v\rangle = 0 \\ G(u,v) &= \langle \boldsymbol{X}_v, \boldsymbol{X}_v\rangle = r^2\cosh^2\theta.\end{aligned}\right\} \tag{24}$$

Thus, if $\boldsymbol{w}$ is a tangent vector to the sphere at the point $\boldsymbol{X}(u,v)$, given in the basis associated to $\boldsymbol{X}(u,v)$ by

$$\boldsymbol{w} = a\boldsymbol{X}_u + b\boldsymbol{X}_v$$

then the square of the length of $\boldsymbol{w}$ is given by

$$|\boldsymbol{w}|^2 = I(\boldsymbol{w}) = Ea^2 + 2Fab + Gb^2 = -a^2r^2 + b^2r^2\cosh^2\theta. \tag{25}$$

Let's determine the timelike (spacelike) helices in this coordinate neighborhood of the Lorentzian sphere $S_1^2(r)$ which make a constant timelike angle θ with the $z-$axis.

Theorem 4 *Let θ be constant hyperbolic angle between the $z-$time axis and the timelike curve $\beta_\lambda(t)$, and assume that $\lambda = \operatorname{sech}\theta$. The parametrization of all timelike helices on the surface of the sphere $S_1^2(r)$ is given by*

$$\beta_\lambda(t) = (\cosh t \cos v(t), \cosh t \sin v(t), \sinh t), \tag{26}$$

where

$$\left.\begin{aligned} v = &-\frac{\sqrt{2}\cosh t\sqrt{-\lambda^2+\operatorname{sech}^2 t}}{\lambda^2+\lambda^2\cosh(2t)-2}\left[-\lambda\sqrt{\lambda^2-1}\sinh^{-1}\left(\frac{\lambda}{\sqrt{\lambda^2-1}}\sinh t\right)\sqrt{\frac{\lambda^2+\lambda^2\cosh(2t)-2}{\lambda^2-1}}\right.\\ &\left.+\sqrt{\lambda^2+\lambda^2\cosh(2t)-2}.\tan^{-1}\left(\frac{\sqrt{2}\sinh t}{\sqrt{\lambda^2+\lambda^2\cosh(2t)-2}}\right)\right]+c, \quad \lambda=\operatorname{sech}\theta \end{aligned}\right\} \tag{27}$$

in which c is constant of integration.

Proof. We may assume that the required curve

$$\begin{aligned}\beta(t) &= \boldsymbol{X}(u(t), v(t))\\ &= (r\cosh u(t)\cos v(t), r\cosh u(t)\sin v(t), r\cosh u(t))\end{aligned}$$

is the image $\boldsymbol{X}$ of the curve $(u(t), v(t))$ of the $uv-$plane.

In this case we have

$$\begin{aligned}\beta'(t) = (&ru'\sinh u\cos v - rv'\cosh u\sin v,\\ &ru'\sinh u\sin v + rv'\cosh u\cos v, u'\cosh u).\end{aligned}$$

Thus, the constant hyperbolic angle between the tangent of the curve and the $z-$time axis is the real number θ satisfying the equality

$$\langle e_3, \beta'(t)\rangle = -|\beta'(t)|\cosh\theta$$

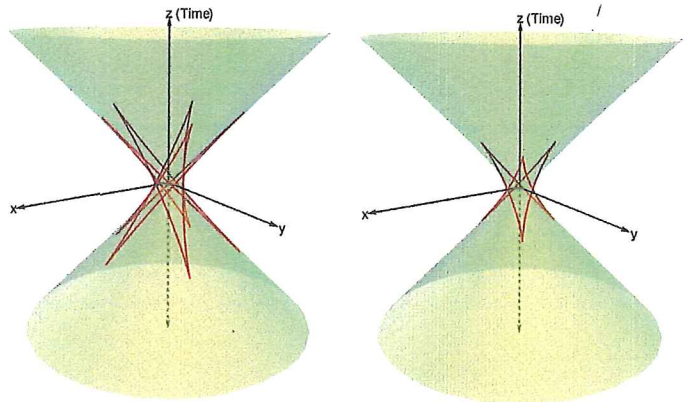
or

$$u'\cosh u = \sqrt{\left|r'\cosh^2 u - u'^2\right|}\cosh\theta. \tag{28}$$

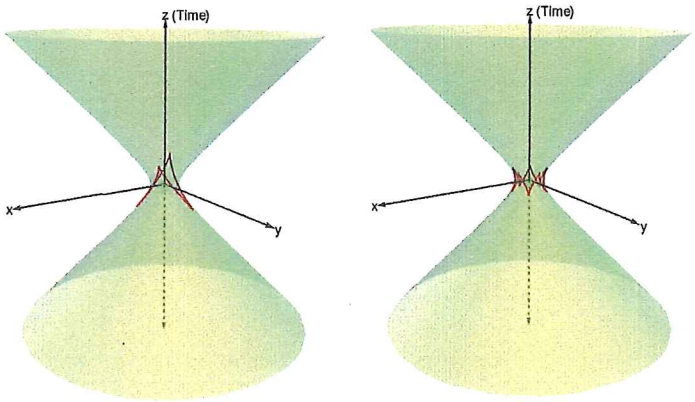
Since in the basis $\{\boldsymbol{X}_u, \boldsymbol{X}_v\}$ the vector $\boldsymbol{\beta}'(t)$ has coordinates (u', v') and the vector $\boldsymbol{X}_u$ has coordinates $(1, 0)$. By (28) we obtain the differential equation

$$\frac{dv}{du} = \sqrt{\operatorname{sech}^2 u - \lambda^2}, \qquad \lambda = \operatorname{sech}\theta.$$

Hence, by integration, we find $v(t)$ given by (27). It is presented some Lorentzian timelike helices for different values of λ^2 in Figure 4. This completes the proof.



(a) $\lambda^2 = \frac{1}{25}$ (b) $\lambda^2 = \frac{1}{9}$



(c) $\lambda^2 = \frac{1}{4}$ (d) $\lambda^2 = \frac{16}{25}$

Figure 4: Lorentzian timelike helices for different values of λ

■

It should be note that the parametrization obtained for the Lorentzian timelike spherical helices is the second important parametrization of the timelike helices in [27] and [28].

Theorem 5 *Let θ be constant timelike angle between the $z-$time axis and spacelike curve $\boldsymbol{\beta}_{\boldsymbol{\lambda}}(t)$, assume that $\lambda = \operatorname{csch}\theta$. The parametrization of all spacelike helices on the surface of the Lorentzian sphere $S_1^2(r)$ are given by*

$$\boldsymbol{\beta}_{\boldsymbol{\lambda}}(t) = (\cosh t \cos v(t), \cosh t \sin v(t), \sinh t), \tag{29}$$

where

$$\left.\begin{aligned} v(t) &= \frac{1}{2+\lambda^2+\lambda^2\cosh(2t)}\sqrt{2}\cosh(t) \\ &\cdot\left[\tan^{-1}\left(\frac{\sqrt{2}\sinh(t)}{\sqrt{2+\lambda^2+\lambda^2\cosh(2t)}}\right)\sqrt{2+\lambda^2+\lambda^2\cosh(2t)}\right. \\ &\left.+\lambda\sqrt{1+\lambda^2}\sin^{-1}\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)\sqrt{\frac{2+\lambda^2+\lambda^2\cosh(2t)}{1+\lambda^2}}\right] \\ &\cdot\sqrt{\lambda^2+\operatorname{sech}^2(t)}+c, \quad \lambda=\operatorname{csch}\theta. \end{aligned}\right\} \tag{30}$$

in which c is constant of integration.

Proof. By 7 the constant timelike angle between the tangent of the curve and the $z-$time axis is the real number θ satisfying the equality

$$|\langle \boldsymbol{e_3}, \boldsymbol{\beta}'(t)\rangle| = |\boldsymbol{\beta}'(t)|\sinh\theta$$

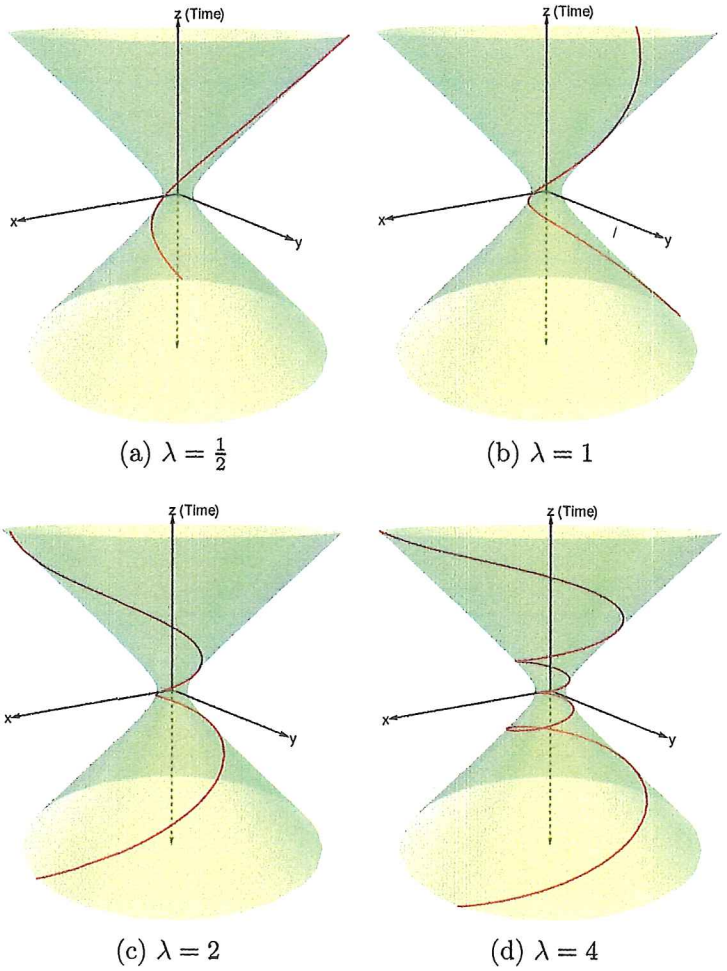
or

$$u'\cosh u = \sqrt{v'^2\cosh^2 u - u'^2}\sinh\theta \tag{31}$$

Thus we have

$$\frac{dv}{du} = \sqrt{\lambda^2+\operatorname{sech}^2 u}, \quad \lambda=\operatorname{csch}\theta. \tag{32}$$

Hence, the solution of the differential equation (32) is the function $v(t)$ given by (30) (Fig. 5)



(a) $\lambda = \frac{1}{2}$ (b) $\lambda = 1$

(c) $\lambda = 2$ (d) $\lambda = 4$

Figure 5: Lorentzian spacelike helices for different values of λ

■

6 Helices on Lightlike Cone Λ^2

We know that the parametrization of the lightlike case Λ^2 is given by

$$\boldsymbol{X}(u,v) = (u\cos v, u\sin v, u), \tag{33}$$

where $0 \leq v \leq 2\pi$ and $u \in \mathbb{R}$. In the parametrization (33), if $v = v(u)$, we find a general curve

$$\boldsymbol{X}(u(t), v(t)) = \gamma(t) = (u\cos v(u), u\sin v(u), u) \tag{34}$$

on the surface of Λ^2.

Now we give the parametrization of the conical helix.

Theorem 6 *The parametrization of the helix on the surface of the lightlike cone Λ^2 is given by*

$$\gamma(t) = \left(e^{\lambda t}\cos t, e^{\lambda t}\sin t, e^{\lambda t}\right), \tag{35}$$

where $\lambda = \sinh\theta$ and θ is timelike angle between the curve $\gamma(t)$ and the $z-$time axis.

Proof. By differentiating the curve (34) with respect to u we have

$$\gamma'(u) = (\cos v(u) - uv'(u)\sin v(u), \sin v(u) + uv'(u)\cos v(u), 1). \tag{36}$$

Thus, the constant timelike angle between the tangent of the curve and the $z-$time axis is the real number θ satisfying the equation

$$|\langle \boldsymbol{e_3}, \gamma'(u)\rangle| = v'(t)\,u(t)\sinh\theta$$

or

$$u'(s) = v'(t)\,u(t)\sinh\theta, \tag{37}$$

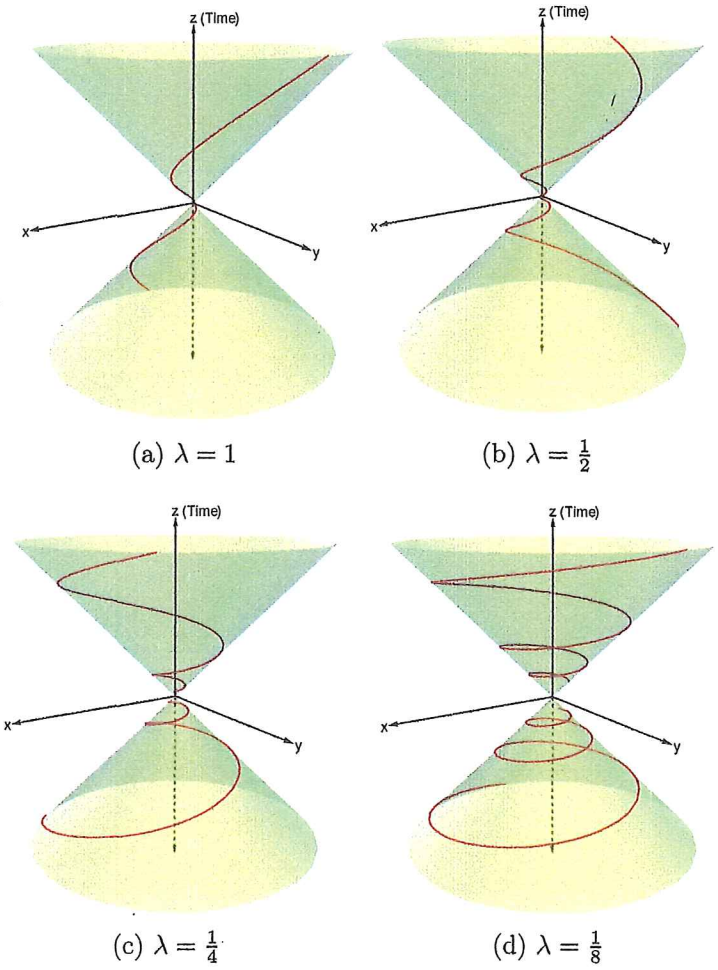
since in the basis $\{\boldsymbol{X_u}, \boldsymbol{X_v}\}$ the vector $\gamma_\lambda(t)$ has coordinates (u', v') and the vector $\boldsymbol{X_u}$ has coordinates $(1, 0)$. By (37) we obtain the differential equation

$$\frac{u'}{u} = \sinh\theta.v' \tag{38}$$

Hence, by integration, we find

$$u = ce^{(\sinh\theta)t}, \tag{39}$$

where c is constant of integration. If we write the values $\lambda = \sinh\theta$, $c = 0$ and (39) in the parametrization, we obtain the helix (35). ■



(a) $\lambda = 1$ (b) $\lambda = \frac{1}{2}$

(c) $\lambda = \frac{1}{4}$ (d) $\lambda = \frac{1}{8}$

Figure 6: Helices on the lightlike cone for different values of λ

7 Conclusion

In this paper, we defined the differential equations of causal helices on the surfaces of the spheres of the Lorentz-Minkowski 3−space $\mathbb{R}_1^3$ and gave the general parametrizations of these curves. In the next study, we will determine the helix curves on the some special surfaces of the space $\mathbb{R}_1^3$ and examine the geometry of these surfaces.

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MATRIX, GROUP, TENSOR, ETC., AND CORRESPONDING MATHEMATICAL QUANTUM THEORY AND ITS APPLICATIONS

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Abstract

Based on the non-commutation $[A,B]=AB-BA=\eta\neq 0$ of matrix, group, tensor and so on, we propose the mathematical quantum theory. If η is imaginary number, it will correspond to the extensive quantum theory. If η is real number, it will be development of quantum theory. Moreover, η may be complex number, etc. We introduce a similar wave function and corresponding operators, various similar quantum results will be derived. Next, we discuss its physical meaning and various applications. Third, based the general matrix we research mathematics of unified gravitational and electromagnetic fields. Fourth, we discuss the space-time equations and the simplest unifying quantum theory and general relativity. Further, we can combine general discrete mathematics.

Keywords: quantum theory; non-commutation; matrix; group; tensor; operator; application; field; unification, general relativity.
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I. Introduction

Quantum mechanics is the best great science in 20 century. Based on well-known Einstein-Podolsky-Rosen (EPR) argument, if two physical quantities are described by non-commuting operators, it will derive the quantum mechanics or its similar results. More generally, it is shown in quantum mechanics that, if the operators corresponding to two physical quantities, say A and B, do not commute, that is, if $AB \neq BA$, then the precise knowledge of one of them precludes such a knowledge of the other [1]. Feynman pointed out: "There are certain situations in which the peculiarities of quantum mechanics can come out in a special way on large scale." In a special situation "quantum mechanics will produce its own characteristic effects on a large or 'macroscopic' scale" [2]. Hey and Walters researched general quantum universe [3]. Freudenthal, et al., developed quantum nanospheres for micron particle image velocimetry [4]. Ahmadi, et al. , developed the theory of quantum electrodynamics (QED) to antimatter, and investigated the fine structure of antihydrogen [5].

In 1998 Santilli shown that the objections against the EPR argument are valid for point-like particles in vacuum (*exterior dynamical systems*), but the same objections are inapplicable (rather than being violated) for extended particles within hyperdense physical media (*interior dynamical systems*) because the latter systems appear to admit an identical classical counterpart when treated with the isotopic branch of hadronic mathematics and mechanics. Santilli reviewed, upgraded and specialized the basic mathematical, physical and chemical methods for the study of the EPR prediction that quantum mechanics is not a complete theory. This includes basic methods [6], apparent proof of the EPR argument [7], and examples and applications, in which the validity of the EPR final statement is the effect that the wavefunction of quantum mechanics does not provide a complete description of the physical reality. The axiom-preserving "completion" of the quantum mechanical wavefunction due to deep wave-overlapping when represented via isomathematics, and shown that it permits an otherwise impossible representation of the attractive force between identical electrons pairs in valence coupling, as well as the representation of *all* characteristics of various physical and chemical systems existing in nature [8]. Moreover, Santill studied the classical determinism of EPR prediction by isomathematics [9].

It is well-known that quantum mechanics has two different forms: wave mechanics and matrix mechanics. We proposed the mechanical wave theory that based on the universal wave-particle duality, along an opposite direction of the developed quantum mechanics by using a method where the wave quantities frequency v and wave length λ are replaced on various mechanical equations, and may derive some new results. From this we derived new operators which represent more physical quantities, and proposed some

nonlinear equations and their solutions which may be probably applied to quantum theory [10].

The operators in quantum mechanics are represented by matrices. In Schrödinger equation energy H is matrix, the wave function is vector. Its development is namely S theory. Conversely, general matrix may be represented by operator, which includes Lorentz transformation (LT) and so on [11]. In this paper, based on the non-commutation of matrix, group, tensor, etc., we propose the mathematical quantum theory and its applications and development.

2. Matrix, etc., and Mathematical Quantum Theory

A form of quantum mechanics is matrix mechanics. Conversely, some mathematical regions possess non-commutative law, for example, 1.general matrix; 2.group and ring; 3. tensor, spinor, torsion, etc., they can form the corresponding mathematical quantum theory. It is probably uncertainty, and has quantum and duality. These aspects can be related to discrete mathematics, including soliton and nonlinearity.

Defining two elements A and B in a set, their matrices or calculations are the non-commutative. Let two matrices A and B, or any two quantities, operations, groups, etc., have commutator:

$$[A,B]=AB-BA=\eta\neq 0. \qquad (1)$$

A vector is a column or a row matrix. Matrix may be used to represent various non- commutative kinds of mathematics such as tensor, spinor, torsion and twistor [12-16], etc. The commutative must be similar quantity in the same set. Graph theory and the hypercycle may be represented by matrix [17].

I. Assume that $\eta=iH$ is an imaginary number, so it corresponds to the extensive quantum theory [18-21], which is related to imaginary quantity. A and B are non-commutative [1], so may derive an extensive conjugation.

II. Assume that η is usual real number. It corresponds to development of quantum theory, and often $i\hbar\to\eta$. We call the mathematical quantum theory. This is a non-commutative algebra. From this we may derive some equations, and build a similar quantum theory. It has the corresponding uncertainty relation:

$$\Delta A\Delta B\geq\frac{1}{2}|[A,B]|=\frac{\eta}{2}. \qquad (2)$$

If [A.B]=0, A and B may derive their simultaneous eigenstate, and two are certainty.

III. Assume that $\eta\to\eta+iH$ is a complex number, it is more general case. Of course, η may also be quaternion, etc.

These are non-commutation of operation. They may also be non-commutation of some quantities. It is Dirac's q-number $[q_i, p_j] = i\hbar\delta_{ij}$ and its extensions. These correspond to non-Abel gauge group and Yang-Mills (YM) field, Utiyama field [22], etc.

3. Physical Meaning and Some Results

In quantum mechanics they are known:

$$p_\alpha = -i\hbar\frac{\partial}{\partial x_\alpha}, E = i\hbar\frac{\partial}{\partial t}. \tag{3}$$

These are related to imaginary numbers.

We research mainly II case, in which η is real number. We introduce a similar wave function and corresponding operators:

$$p_\alpha = -\eta\frac{\partial}{\partial x_\alpha}, \; E = \eta\frac{\partial}{\partial t}. \tag{4}$$

The wave-particle duality and various similar quantum results will be derived. For equation $E\psi = \eta\frac{\partial\psi}{\partial t}$, its solution is $\psi = C_0 e^{(E/\eta)t}$.

The non-relativistic energy is:

$$E = \frac{p^2}{2m} + V = \frac{\eta^2}{2m}\Delta + V. \tag{5}$$

Such the corresponding Schrödinger equation is:

$$\eta\frac{\partial\psi}{\partial t} = (\frac{1}{2m}\eta^2\nabla^2 + V)\psi. \tag{6}$$

The time-independent Schrödinger equation is:

$$\Delta\psi = \frac{2m}{\eta^2}(E - V)\psi. \tag{7}$$

The relativistic energy is:

$$E^2 = p^2c^2 + m^2c^4. \tag{8}$$

Such the corresponding Klein-Gordon (KG) equation is:

$$[c^2\eta^2\frac{\partial^2}{\partial x_\alpha^2} - \eta^2\frac{\partial^2}{\partial t^2}]\psi + m^2c^4\psi = 0. \tag{9}$$

$$[\frac{\partial^2}{\partial x_\mu^2} + \frac{(mc)^2}{\eta^2}]\psi = 0. \tag{10}$$

It describes bosons with spins J=0, ... n, which include 2 or 2n quarks and have not the magnetic moment.

The corresponding Dirac equation is:

$$\gamma_\mu \partial_\mu \psi + \frac{mc}{\eta}\psi = 0. \quad (11)$$

It describes fermions with spins J=1/2, ... n/2, which include 3 or 3+2n quarks and have the magnetic moment. The spins at a certain degree can unify fermion and boson J=n/2.

These are all real numbers and corresponding equations.

The spins, the angular momentum, the generators in group and so on are all non-commutation [23]. Further, they may extend to the momentum-energy representations [11].

If A and B are two conjugate quantities, it will form Heisenberg equation:

dF/dt=[F,H]. (12)

and Poisson bracket. Three symmetric equations are [24]:

$$\frac{dF}{dt} = \frac{1}{i\hbar}[F,H], \quad (13)$$

$$\frac{dF}{dr} = \frac{1}{i\hbar}[F,P], \quad (14)$$

$$\frac{dF}{d\varphi} = \frac{1}{i\hbar}[F,M_z]. \quad (15)$$

For III case,

$$p_\alpha = -(\eta + iH)\frac{\partial}{\partial x_\alpha}, \; E = (\eta + iH)\frac{\partial}{\partial t}. \quad (16)$$

Schrödinger equation is:

$$(\eta + iH)\frac{\partial \psi}{\partial t} = \frac{1}{2m}[(\eta + iH)^2\nabla^2 + V]\psi. \quad (17)$$

Klein-Gordon (KG) equation is:

$$[c^2(\eta + iH)^2\frac{\partial^2}{\partial x_\alpha^2} - (\eta + iH)^2\frac{\partial^2}{\partial t^2}]\psi + m^2c^4\psi = 0. \quad (18)$$

Dirac equation is:

$$\gamma_\mu(\eta + iH)\partial_\mu\psi + mc\psi = 0. \quad (19)$$

The symmetry and its violation, energy conservation and various basic principles of physics [25] are universal.

4. Various Applications

1. One-dimensional time-independent equation is:

$$\frac{d^2\psi}{dx^2}-\frac{2m}{\eta^2}(E-V)\psi=0. \qquad (20)$$

Here $\lambda_{1,2}=\pm\sqrt{2m(E-V)/\eta^2}=\pm\lambda$. Its solution is: $\psi=Ce^{\lambda x}+C'e^{-\lambda x}\propto Ach\lambda x$. For E>V they are all positive values. The hyperbolic function may be related to the negative matter [26-33].

If V>E, $k_{1,2}=\pm i\sqrt{2m(V-E)/\eta^2}=\pm ik$, the solution of Eq.(20) is $\psi=A\sin(kx+\delta)$. For infinite square well V=0(0<x<a) and V=∞(x<0, x>a) [23], sinka=0, so $ka=n\pi$,

$$2m(V-E_n)a^2=\eta^2n^2\pi^2, \text{ and } E_n=V-\frac{\eta^2n^2\pi^2}{2ma^2}. \qquad (21)$$

Its energy levels are quantized, except for infinity potential V, which is possibly related to renormalization.

2. For a central symmetric field

$$E\psi=(\frac{\eta^2}{2m}\nabla^2+V)\psi. \qquad (22)$$

When V>E, the equation is:

$$\Delta\psi+\frac{2m}{\eta^2}(V-E)\psi=\Delta\psi+k^2\psi=0, \qquad (23)$$

which is Helmholtz equation, i.e., wave equation:

$$(\frac{\eta^2}{2m}\frac{1}{r}\frac{\partial^2}{\partial r^2}r-\frac{M^2}{2mr^2}+V)\psi=E\psi. \qquad (24)$$

Its result compares quantum mechanics only the sign of first term is opposite. Second term is the centrifugal potential. When V=E, it is Laplace equation.

In spherical coordinates, by general method let $\psi=R(r)Y(\theta,\varphi)=R(r)\Theta(\theta)\Phi(\varphi)$, in which R(r) is the spherical Bessel equation, $\Theta(\theta)$ is the associated Legendre equation, and

$$\Phi(\varphi)=A\cos m\varphi+B\sin m\varphi. \qquad (25)$$

$\Theta(x)=(1-x^2)^{m/2}P_j^m(x)$ is the associated Legendre function. We discussed various decays of particles and some known decay formulas. Many important decays of particles (0^{-+} mesons, $(1/2)^+$ baryons and $\mu^{\pm}$) and some known decays of resonances can be generally described by the square of some types of the associated Legendre functions P_l^l and P_{l+1}^l. Its universal formula is:

$$\Gamma=\frac{G^2}{4\pi N}m_i^{\alpha}\mid P_l^{|m|}(x)\mid^2, \qquad (26)$$

in which x is the ratio of the mass-sum of the final state and the mass of the initial state of particle. It is combined with the Δ I=1/2 rule, then the agreements are very good. The decay formulas of the similar decay modes are the same. For the same types of particles, the coupling constants are kept to be invariant, and only six constants are independent. The simplest dynamical basis is probably that the decay formulas are the square of the solution of free field equation of the decay particles in the momentum coordinate system [34,35]. Further, we discussed some general decays and their rules, and apply the decay formulas to some massive hadrons [35]. Various baryons pass through weak interaction to become proton. Various mesons pass through weak interaction to become electron or photon.

Let x=kr and $R(r)=\sqrt{\frac{1}{x}}y(x)$,

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+[x^2-(j+\frac{1}{2})^2]y=0. \qquad (27)$$

It is j+(1/2) order Bessel equation. Its solution is $y=Ax^{j+(1/2)}+Bx^{-j-(1/2)}$.

In column coordinates, by general method let $\psi=R(\rho)Z(z)\Phi(\varphi)$.

$$Z''-\mu Z=0. \qquad (28)$$

Its solution is $Z(z)=Ce^{\sqrt{\mu}z}+De^{-\sqrt{\mu}z}$ for μ>0. For μ<0, let $-\mu=h^2$, solution is:

$$Z(z)=C\cos(hz)+D\sin(hz). \qquad (29)$$

Eq.(23) becomes to

$$\rho^2\frac{d^2R}{d\rho^2}+\rho\frac{dR}{d\rho}+[\rho^2(k^2-h^2)-m^2]R=0. \qquad (30)$$

Let $x=\sqrt{k^2-h^2}\rho$, Eq.(30) becomes to

$$x^2\frac{d^2R}{dx^2}+x\frac{dR}{dx}+[x^2-m^2]R=0. \qquad (31)$$

R(x) is m order Bessel equation. Its solution is $R=Ex^m+Fx^{-m}$. Eqs. (27) and (31) should correspond to fermions and bosons, respectively. Here j(j+1) and m are all quantized. Both correspond to angular and magnetic quantum numbers. Quantization derives the stationary states, from this leads to quantum transitions.

The solutions of more general trigonometric functions (corresponding to volatility) are quantized.

3. In electromagnetic field $E=\frac{1}{2m}(p-\frac{q}{c}A)^2+q\varphi$,

$$\eta\frac{\partial\psi}{\partial t}=\frac{1}{2m}[\eta^2\nabla^2-\eta\frac{q}{c}(\nabla A+A\nabla)+(\frac{q}{c}A)^2]\psi+q\varphi\psi\,. \quad (32)$$

From this may obtain the magnetic moment $\mu=\frac{e\eta}{2mc}$.

Another important concept of quantum mechanics is phase. Phase differences have the magnetic AB (Aharonov-Bohm) effect:

$$\delta\phi=(q/c\eta)\oint A(r)dr\,, \quad (33)$$

and the electronic AB effect:

$$\delta\phi=(q\tau/\eta)\Delta\varphi\,. \quad (34)$$

A big progress of the extensive quantum theory is non-locality and quantum entanglement.

$$\{A,B\}=AB+BA=\eta'\neq 0\,. \quad (35)$$

If {A, B}=AB+BA=0 is anti-commutative, so [A,B]=2AB. Probably, it corresponds to BE statistics, BEC and bosons.

We combine geometric optics (corresponds to particle) and wave optics (corresponds to wave), $\Delta\lambda\Delta r=\eta_1$ and $\Delta\nu\Delta t=\eta_2$. The wave-particle duality develops to the wave-timespace duality. Time-space must be related to relativity.

The matrix can represent any set of discrete elements. It and the mathematical quantum theory can describe from quantized atoms, molecules, ions to discrete neurons, genes, cells, individuals, and typical DNA in biology, and more general quantum physics, quantum chemistry, quantum biology, quantum sociology and quantum social science.

Matrix may be applied to the balancing equations in chemistry, engineering, ecology, network flow, business, management science [36], etc. In economics the matrix represents the Leontief input-output model [37] and the social accounting matrix (SAM) in model of social-economic system. It is the matrix equations for population planning, and extended to ecology and infectious diseases, nutrition and matrix pharmacokinetics [36] and various linear programming models.

The network theory belongs to the general graph, and graph theory can be represented by matrix, and be quantized. It is completely similar to the particle lattice theory.

5. Mathematics of Unified Gravitational and Electromagnetic Fields

For tensors $A_iB_j=C_{ij}$, so

$$A_{ij}B_{kl}-B_{kl}A_{ij}=C_{ijkl}-C_{klij}\,. \quad (36)$$

The commutation of tensor is converted into the commutation of index. The commutation and the anti-commutation correspond to the scalar product ***AB=BA*** and the vector product $A\times B=-B\times A$ of two vectors. They are related to the simplest electric and magnetic fields.

One dimensional harmonic oscillator (x_{mn}) is a symmetric matrix [38]. Conversely, the symmetric matrix corresponds to the generalized space-time, and the gravitational field. Momentum (p_{mn}) is an antisymmetric matrix [38]. Conversely, the antisymmetric matrix corresponds to the generalized momentum, and the electromagnetic field. Energy (H_{mn}) is a diagonal matrix [38]. Conversely, the diagonal matrix corresponds to the generalized energy. And space-time and momentum-energy are conjugation, and uncertainty each other.

More generally, the symmetric tensor corresponds to the gravitational field of general relativity, and the antisymmetric tensor corresponds to electrodynamics and electromagnetic fields. Such the unification of gravitational-electromagnetic fields corresponds to general matrix, which includes possibly torsion.

In general relativity,

$$g_{ik}=g_{ki},\Gamma^{l}_{ik}=\Gamma^{l}_{ki},R_{ik}=R_{ki} \tag{37}$$

all are symmetric. But, electrodynamics

$$F_{ik}=-F_{ki} \tag{38}$$

are antisymmetric. This has some similar characters with torsion.

The difference of translation in curve coordinates is $\Gamma^{i}_{kl}A^{k}dx^{l}$, and derives the Christoffel symbols, tensor, torsion, etc. This is probably related to loop, quantum gravity, string and so on.

Therefore, mathematics of unified gravitational and electromagnetic fields should be general

$$H_{\mu\nu}=R_{\mu\nu}+F_{\mu\nu}. \tag{39}$$

In 16 elements of four-dimensional matrix, the symmetric part are gravitational field, the antisymmetric part are electromagnetic field.

$$H_{\mu\nu}=R_{\mu\nu}+F_{\mu\nu}=\begin{pmatrix} R_{11} & R_{12}+H_3 & R_{13}-H_2 & R_{14}-iE_1 \\ R_{12}-H_3 & R_{22} & R_{23}+H_1 & R_{24}-iE_2 \\ R_{13}+H_2 & R_{23}-H_1 & R_{33} & R_{34}-iE_3 \\ R_{14}+iE_1 & R_{24}+iE_2 & R_{34}+iE_3 & R_{44} \end{pmatrix}. \tag{40}$$

General space-time $ds^2=g_{\mu\nu}dx_{\mu}dx_{\nu}$,

$$g_{ik} = \frac{1}{2}(g_{ik} + g_{ki}) + \frac{1}{2}(g_{ik} - g_{ki}). \qquad (41)$$

It may be divided into symmetric and antisymmetric parts. For gravitational field $g_{\mu\nu} = g_{\nu\mu}$, the electromagnetic field $g_{\mu\nu} = -g_{\nu\mu}$. Particles have spin and spinor, and some fields have rotation as rot ***A***.

For $(x_\alpha; x_0 = ct)$, Lorentz transformation (LT) is symmetric matrix:

$$\begin{pmatrix} \gamma & 0 & 0 & \gamma\frac{v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\frac{v}{c} & 0 & 0 & \gamma \end{pmatrix}. \qquad (42)$$

Based on general relativity $s^2 = g_{\mu\nu}x_\mu x_\nu$, we proposed the quantum equations of general relativity are [11]:

$$s^2\psi = g_{\mu\nu}x_\mu x_\nu\psi = -\hbar^2 g_{\mu\nu}\frac{\partial^2}{\partial p_\mu \partial p_\nu}\psi. \qquad (43)$$

$$g_{\mu\nu}\frac{\partial^2}{\partial p_\mu \partial p_\nu}\psi + \frac{s^2}{\hbar^2}\psi = 0. \qquad (44)$$

$$(g_{\alpha\alpha}\frac{\partial^2}{\partial p_\alpha^2} + 2g_{\alpha 0}\frac{c\partial^2}{\partial p_\alpha \partial E} - g_{00}\frac{c^2\partial^2}{\partial E^2})\psi + \frac{s^2}{\hbar^2}\psi = 0. \qquad (45)$$

The operator of energy-momentum tensor corresponds to the second derivative:

$$T_{\mu\nu} = ap_\mu p_\nu = -a\hbar^2\frac{\partial^2}{\partial x_\mu \partial x_\nu}. \qquad (46)$$

It corresponds to gravitational field.

The unity of quantum theory and general relativity is:

$$H_{\mu\nu}(R_{\mu\nu})\psi = \kappa T_{\mu\nu}\psi. \qquad (47)$$

It is quantum non-symmetric tensor. The left side and right side of Eq.(47) are the eigenmatrices and eigenvalues of the eigen equation. Both sides are all operators.

Mathematics of relativity are vector, tensor, and torsion, they are all spatial properties. General $\Gamma^\lambda_{\mu\nu} = \frac{1}{2}(\Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\nu\mu}) + \frac{1}{2}(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu})$ may be divided into symmetric and antisymmetric parts, both have respectively with 40

degrees of freedom and 24 degrees of freedom, in which antisymmetric part $T^{\lambda}_{\mu\nu}=\frac{1}{2}(\Gamma^{\lambda}_{\mu\nu}-\Gamma^{\lambda}_{\nu\mu})$ is called torsion. It corresponds to spin tensor [39]. Quantum theory searches the spinor with two-component, which is related to Pauli matrix.

6. Quantum Time-Space Equations and Their Developments

The unifying quantum mechanics and general relativity is always an important problem in modern physics. Relativity mainly studies space-time, quantum mechanics mainly studies microscopic particles. Based on the time-space operators of energy-momentum representation in quantum mechanics, we discussed the space-time operators and their generalization, and proposed some operator equations of general relativity and special relativity. We researched some applications of this method, in particular, the lifetime formulas of particles are obtained from the time equation, and they agree better with the experimental data. It is the simplest unifying quantum theory and general relativity, and corresponds to the extensive quantum theory [18-21], and may overcome the singularity problem in general relativity. In a word, this is the combination and unification on quantum mechanics and general relativity [11].

In momentum-energy operators representation there are the quantum equations [24,40]:

$$x_{\mu}\psi=i\hbar\frac{\partial}{\partial p_{\mu}}\psi. \tag{48}$$

They include the time operator equation in energy representation:

$$t\psi=-i\hbar\frac{\partial}{\partial E}\psi. \tag{49}$$

and the space operator equation:

$$r\psi=i\hbar\frac{\partial}{\partial p}\psi. \tag{50}$$

The corresponding mathematical quantum equations in momentum-energy operator representation are:

$$x_{\mu}\psi=\eta\frac{\partial}{\partial p_{\mu}}\psi. \tag{51}$$

We discussed generally any transitions among $p_{\mu}(E),x_{\mu}(t)$ and corresponding operators, and researched their meanings. As result, many aspects of quantum mechanics and general relativity can be opened up. For example, we may combine three roads to quantum gravity: superstring, loop

quantum gravity and so on [41].

Lorentz transformation (LT) with space-time relativity is representation:

$$x'\psi = \frac{i\hbar(\frac{\partial}{\partial p} + v\frac{\partial}{\partial E})}{\sqrt{1-(v/c)^2}}\psi, \quad t'\psi = -\frac{i\hbar(\frac{\partial}{\partial E} + \frac{v}{c^2}\frac{\partial}{\partial p})}{\sqrt{1-(v/c)^2}}\psi. \quad (52)$$

Since space and motion may affect time, the time function t(r,v) is obtained. Further, it may combine the extensive special relativity [34,42] and the extensive general relativity [43]. Both are unified into four vectors. If they cannot be unified, time is directional, and corresponds to the semigroup and irreversibility of time [44], and to the non-commutation of quantum mechanics, in particular, the non-commutation of time operator $t_1 t_2 \neq t_2 t_1$, in which

$$t_1 t_2 = -\hbar^2(\frac{\partial}{\partial E_1}\frac{\partial}{\partial E_2}). \quad (53)$$

It is known that the non-commutation on order of differential variables. More generally, this is the non-commutation on order of integrals and operators and their calculus. It is related to operator of LT, and to the non-commutation of time-space.

Developments of Eq.(47) may be: 1)Time-space quantization:

$$dx_\mu\psi = i\hbar\frac{\partial}{\partial p_\mu}d\psi, \quad (54)$$

derives wave function quantization.

2) $$dx_\mu\psi = i\hbar d[\frac{\partial}{\partial p_\mu}\psi]. \quad (55)$$

3) $$dx_\mu\psi = id[\hbar\frac{\partial}{\partial p_\mu}\psi] = i[(d\hbar)\frac{\partial}{\partial p_\mu} + \hbar d(\frac{\partial}{\partial p_\mu})]\psi. \quad (56)$$

$d\hbar$ may obtain the extensive quantum theory [18-21] which is combined, and corresponding wave function should be extended.

Various operators $\frac{\partial}{\partial t}, \frac{\partial}{\partial x_\alpha}$ become to $\frac{i}{\hbar}E, -\frac{i}{\hbar}p_\alpha$, for example,

$$v = \frac{dr}{dt} = i\hbar\frac{d}{dt}\frac{\partial}{\partial p}, \quad (57)$$

$$a = \frac{d^2r}{dt^2} = i\hbar\frac{d^2}{dt^2}\frac{\partial}{\partial p}, \quad (58)$$

$$F = \frac{dp}{dt} = -i\hbar\frac{d}{dt}\frac{\partial}{\partial x}. \quad (59)$$

Gravitation $F=-\frac{Gm_1m_2}{r^2}$ may become to:

$$F\psi=\frac{Gm_1m_2}{\hbar^2\dfrac{\partial^2}{\partial p^2}}\psi, \tag{60}$$

or $$F\psi=-\frac{GE_1E_2}{c^4r^2}\psi=\frac{G\hbar^2}{c^4r^2}\frac{\partial^2}{\partial t_1\partial t_2}\psi, \tag{61}$$

or $$F\psi=-\frac{G}{c^4}\frac{\dfrac{\partial^2}{\partial t_1\partial t_2}}{\dfrac{\partial^2}{\partial p^2}}\psi. \tag{62}$$

Further, by this method we may search electromagnetic field, and Lorentz equation and Maxwell equations, and strong and weak interactions, etc. Various formulas may be extended by $i\hbar\to\eta$ and Eq.(51). The four basic interactions as operators should be the non-commutation.

Present quantum mechanics is mainly linear. Quantum mechanics applied gravitation must be analogue with gravitational field, and develops nonlinear theory [34,45]. Their forms may be analogy of De Broglie-Bohm nonlinear quantum mechanics [46,47], or of Heisenberg nonlinear theory and equations [48,49] developed Dirac equations. We develop time-space (t-r) and energy-momentum (E-p), etc., to some nonlinear theories. Or they are only:

$$\gamma_\mu\frac{\partial}{\partial p_\mu}\psi+l_0^3\psi^3=0. \tag{63}$$

$$\frac{d}{dE}\psi+l_0^3\psi^3=0. \tag{64}$$

Its solution is:

$$\psi=C(2l_0^3E)^{-1/2},\ \therefore\psi^2=\frac{C_0}{2l_0^3E}. \tag{65}$$

One side, we developed the electromagnetic general relativity [43], and may obtained Rutherford scattering. Other side, we researched a possible nonlinear quantum theory [34,45], from which introduces F corresponds to $g_{\mu\nu}$, and Γ corresponds to potential V. Combining quantum optics and statistics optics, we may discuss quantum relativity and statistical relativity, etc. More general, linear operators develop into nonlinear operators and nonlinear quantum mechanics. Further, linear algebra develops into nonlinear algebra and nonlinear space, etc.

Based on various known unified theories of interactions in particle physics, we consider that strong and weak interactions as short-range should more be unified. Except different action ranges their main character is: strong interactions are attraction each other, and weak interactions are mutual repulsion and derive decay. We proposed a possible method on their unification, whose coupling constants are negative and positive, respectively, and proposed a figure on the unification of the four basic interactions in three-dimensional space (Fig.1), and searched some possible tests and predictions, for example, strong-weak interactions transform each other, some waves may be produced. Further, based on the simplest unified gauge group GL(6,C) of four-interactions, a possible form of Lagrangian is researched. Some relations and equations of different interactions are discussed [50].

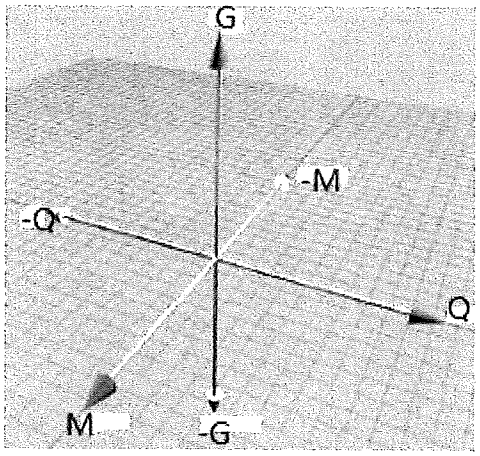

Fig. 1. The unification on the four basic interactions in three-dimensional space

This method can be extended to the mechanical wave theory [10], and may obtain conjugate equation, mass-life formula and so on.

7. Discussion

Von Neumann published a book *Mathematical Foundations of Quantum Mechanics* [51]. Here we research the mathematical quantum theory and its applications and extensions.

Time is currently a scalar, there is only a gradient:

$$\nabla t = \frac{\partial t}{\partial x_\alpha}\vec{i} = -\frac{i}{\hbar}\vec{p}_\alpha t . \qquad (66)$$

Time extends to vector ***T***, which has a divergence:

$$\nabla\vec{T} = \frac{\partial T_\alpha}{\partial x_\alpha} = -\frac{i}{\hbar} p_\alpha T_\alpha, \tag{67}$$

and a curl:

$$rot\vec{T} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ T_x & T_y & T_z \\ i & j & k \end{bmatrix} = -\frac{i}{\hbar} \begin{bmatrix} p_x & p_y & p_z \\ T_x & T_y & T_z \\ i & j & k \end{bmatrix}. \tag{68}$$

The life of everyone is a scalar. But, if life develops to the three-dimensional truth-goodness-beauty vector [52], it will able be strong or weak in different dimensions.

When the time is the direction, we should develop the corresponding equation. In this case the time is the vector, and corresponds to the three-dimensional time.

$$-\hbar^2 \Delta\psi = \vec{t}\,\psi, \tag{69}$$

More generally, time and so on can be the matrix. Further, time is developed into tensor and spinor, etc. At this case, the scalar field, the vector field, etc., seem to be related to the operator calculus.

In statistical mechanics and entropy:

$$S\psi = \frac{E}{T}\psi = i\hbar \frac{1}{T}\frac{\partial}{\partial t}\psi. \tag{70}$$

Here the simplest is h→h/T.

In Eq.(49) the life changes with energy. There is oscillation, such as resonance or non-resonance, and rotation. It can be used to study life, spectral and quantization. V(E) is the factor that affects the lifetime, space-time and its structure, and may be the external cause and external potential.

The infinite potential well corresponds to the limited life of everyone, and different people have different energy levels. It can be combined with the different needs of Maslow [53].

Dirac's q-number seems to may combine the extension of Dirac extraction [54], and the negative matter as development of Dirac negative energy [26-33]. Further, we can combine general discrete mathematics [55].

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THE QUANTIZATION EFFECT AND THE HARMONIC OSCILLATOR

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Abstract

In this paper, the ray theory and the relativistic equations of motion of a particle are used to obtain exactly the Planck and De Broglie relations. Then, we study the harmonic oscillator to show how quantization of the mechanical energy results from the Ehrenfest's condition. We applied the theory to the photoelectric effect which can now be explained as a wave effect without photon.

Keywords: Quantum mechanics; Photoelectric effect; harmonic oscillator

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1.INTRODUCTION

At the end of the nineteenth century, classical mechanics and electromagnetism were sufficient theories to describe the interaction between matter and radiation. At the turn of the twentieth century, classical physics was seriously challenged when it fails to explain several microscopic phenomena such as blackbody radiation, atom spectroscopy and the photoelectric effect.

A breakthrough came in 1900 where Planck introduced the concept of the quantum of energy to explain the blackbody radiation. He argues that the interaction between matter and radiation occurs only in integer multiple of $\hbar\omega$ where $\hbar$ is the so called Planck's constant. In 1905, Einstein used the Planck quantum concept to explain the photoelectric effect confirming the particle aspect of waves. Later, De Broglie in 1923 introduced in his thesis a new concept, namely materiel particles display wave-like behavior opening the way to the famous concept of wave-particle duality of matter and radiation.

It is important to point out that classical mechanics is a point theory where the vector $\mathbf{R}(t)$ defines the position of a point materiel particle while for a wave, we need two parameters, the amplitude and the phase of the wave. Therefore, particles and waves seem to be mutually exclusive. We will show in this paper, that it is possible to reconcile in the context of classical physics the wave-particle duality and the quantization of matter and radiation as soon as we have identified the origin of the problem. The principal difficulty to reconcile classical mechanics with electromagnetism comes from the fact that material particles are considered to be point particles instead of wave packets. We refer to our book (1,p.261) for a review of the wave packet concept for the interested reader.

The fact that a particle is not a point has been proved by electron scattering experiments done by Hofstadter (2) in 1956 who proved that all elementary particles have a measurable finite size, an internal charge distribution and can deform themselves in interaction. Hofstadter received the Nobel Prize in 1961 for his important discovery.

With a different approach, studies of the structure of an elementary particle were done by H. Dehmelt (3) from 1976 to 1990. In these experiments, an electron, almost at rest, was isolated and closely confined in a ultrahigh-vacuum Penning trap. These experiments permitted to measure

the dimensionless gyromagnetic factor g with incredible accuracy and also yield a new experimental value for the electron radius $R_e \approx 10^{-22}$ m which is smaller that the known classical radius of the electron $R_e \approx 10^{-15}$ m.

Schrödinger, one of the founders of quantum mechanics, initially considered quantum waves as real mechanical waves. Finally, Schrödinger joined the probabilistic interpretation of the nature of waves in his paper (4) in 1950 where the quantum waves are used as a mathematical concept to calculate the probability of finding the particle. We recall that in quantum mechanics, the wave packet concept has nothing to do with the constitution of an elementary particle since the wave packet concept in quantum mechanics cannot be taken seriously because of the dispersion of the wave packet which rapidly disappears with time and space. Therefore, Schrödinger as many physicists has objected to Copenhagen interpretation because it treats the wave function as just an auxiliary mathematical tool.

Several physicists (4-6) rejected the concept of particle despite experimental evidences which is a bad practice with regards to the scientific truth. The failure by the scientific community to recognize the existence and the finite size of particles independently of any measurements is at the origin of all the difficulties encountered especially in quantum mechanics to give a coherent description of the physical reality within classical physics.

We can find in the literature only a few model of the electron morphology, the first one by Parson (7) in 1915 where the author proposed a model for the electron with a ring-shaped geometry where an elementary charge moves around the ring with the speed of light generating a magnetic field.

Several physicists supported Parson's ring electron model such that Compton (8) who wrote a paper showing that his new found Compton effect was better explained with Parson's model than with the classical model that depicted the electron as a sphere. All these studies were compiled by Allen (9) in 1918.

We have to wait until 1985, to see the ring electron model revisited by Bostick (10) where the angular momentum of the electron or spin has for value $\hbar/2 = \alpha\hbar\, ln(R/r)$ where $2r$ is the diameter of the toroidal shell and R the radius of the toroid and α the fine-structure constant. A ring electron model was also presented by Bergman (12) in 1990 where the spin has now for value $\hbar/2 = \alpha\hbar\, ln(8R/r)/2\pi$. In this last model, the instability of the electron is canceled by the presence of a magnetic Pinch effect where the

magnetic pressure compensates exactly the electrostatic pressure. Several researchers, such as Jennison (11), Kanarev (13) and Lucas (16) proposed similar models. More recently Consa (17) in 2018 proposed a helical solenoid electron model where the electron has a toroidal moment, a feature that is not predicted by quantum mechanics.

A more complete wave model of the electron was developed by Mills (14) in 2003 where the classical wave equation is solved with the constraint that the bound state electron cannot radiate energy. With the assumption that physical laws including Maxwell's equation apply to bound electrons, the hydrogen atom was solved exactly from first principles. The remarkable agreement across the spectrum of experimental results indicates that this is the correct model of the hydrogen atom. In a second paper (15), the physical approach was applied to multi-electron atoms that were solved exactly. The general solutions for one-through twenty-electron atoms are given. The predictions of the ionization energies are in remarkable agreement with the experimental values known for 400 atoms and ions.

From 1960 to 1969, Bourret, Marshall and Boyer (18-21) published a series of papers dealing with a classical definition of the blackbody radiation by assuming that vacuum space is filled with a random zero-point field. All these papers are based on conceiving quantum mechanics as a stochastic process which constitute the essence of the theory known under the name of stochastic electrodynamics (S.E.D).

The quantum electrodynamics theory is plagued with two difficulties related to the existence of infinite quantities and divergent integrals when the variable of integration tends to infinity. Boudet (22,23,1,p.530) discovered a way of carrying through calculations making it possible to avoid infinities in quantum electrodynamics. Boudet developed a method explaining the Lamb shift in an hydrogen atom using a finite-electrodynamics model. He demonstrated how the Lamb shift formula can be obtained without divergent integrals by using classical calculations with Maxwell's equations. The Planck's constant $\hbar$ is introduced in the expression of the current at the end of the calculation for dimensional purpose only.

We can also quote an excellent review paper (24) where the authors show that certain quantum phenomena can be encountered in classical physics. Many physicists think that we cannot use concepts of classical physics to understand and explain quantum mechanics. In contrast with those opin-

ions, we defend in this paper, the point of view that quantum mechanics can be explained as a wave effect. Indeed, most of the quantum phenomena can be explained in the context of classical physics where the Planck's constant $\hbar$ is used as a dimensional constant to relate units. However, the photoelectric effect remains an exception since it is the most difficult effect to be explained in terms of classical physics. The present paper will address this problem and we will prove that this effect can be perfectly explained as a wave effect.

2.THE PLANCK AND DE BROGLIE RELATIONS

It is not well known that the famous Planck and De Broglie relations can be derived from first principles in a rigorous manner as demonstrated in several papers (25-28). We summarize the demonstration to show the connection with the following study of the harmonic oscillator. In vacuum, the motion of a particle, with a rest mass m_0, submitted to a given force $\mathbf{F}$ is described by the relativistic motion equation

$$\frac{d}{dt}(m_0\gamma\mathbf{V}) = \mathbf{F} \tag{1}$$

with the definition $\gamma^2 = (1-\mathbf{V}^2/c^2)^{-1}$. We can rewrite the motion equation in a dyadic form

$$\overleftrightarrow{\mathbf{M}} \cdot \frac{d\mathbf{V}}{dt} = \mathbf{F} \quad \Rightarrow \quad \frac{d\mathbf{V}}{dt} = [\overleftrightarrow{\mathbf{M}}]^{-1} \cdot \mathbf{F} \tag{2}$$

where the direct and inverse mass dyads have for definitions:

$$\overleftrightarrow{\mathbf{M}} = m_0\gamma(\overleftrightarrow{\mathbf{I}} + \frac{\gamma^2}{c^2}\mathbf{VV}) \quad \Rightarrow \quad [\overleftrightarrow{\mathbf{M}}]^{-1} = \frac{1}{m_0\gamma}(\overleftrightarrow{\mathbf{I}} - \frac{1}{c^2}\mathbf{VV}) \tag{3}$$

where $\overleftrightarrow{\mathbf{I}}$ is the unit dyadic.

The fact that the velocity of a material particle submitted to a constant force does not increase linearly with time means that the particle is submitted to a braking force from the medium. In solid-state physics, this braking force is produced by the lattice periodic field. By analogy with the effective mass concept in solids, we can assume that the dyadic mass of the particle

results from the motion of the particle in a waving vacuum. Therefore, the dependence of its mass upon velocity can be explained in the framework of classical mechanics. To do so, we consider the motion equation of a particle in a solid from a wave approach where the internal force which originates from the lattice periodic field remains hidden in the electron effective mass. The particle is now assumed to be a wave packet which obeys a dispersion law $\omega(t) = \omega[\mathbf{k}(t)]$ in the medium. The equation of motion can be written in a form depending only on the external force:

$$\overleftrightarrow{\mathbf{M}} \cdot \frac{d\mathbf{V}}{dt} = \mathbf{F}_e \tag{4}$$

where $\overleftrightarrow{\mathbf{M}}$ is now the effective mass of the particle. To demonstrate the above equation, we assume the existence of a dispersion law $\omega(t) = \omega[\mathbf{k}(t)]$ in the medium and we use the formula of ray theory:

$$\mathbf{V} = \frac{d\mathbf{R}}{dt} = \frac{\partial \omega}{\partial \mathbf{k}} = \frac{1}{\hbar}\frac{\partial E}{\partial \mathbf{k}} \quad \Rightarrow \quad \frac{d\mathbf{V}}{dt} = \frac{1}{\hbar}\nabla_k \nabla_k E \cdot \frac{d\mathbf{k}}{dt} \tag{5}$$

and the quantization law $E(\mathbf{k}) = \hbar\omega(\mathbf{k})$ which will be examined hereafter. We now multiply equation 4 by $\mathbf{V}$ in order to get the relation:

$$\mathbf{V} \cdot \mathbf{F}_e = \frac{dE}{dt} = \frac{\partial E}{\partial \mathbf{k}} \cdot \frac{d\mathbf{k}}{dt} = \hbar \mathbf{V} \cdot \frac{d\mathbf{k}}{dt} \tag{6}$$

The preceding relation can be used to recover the well-known formulas of solid state physics:

$$\mathbf{F}_e = \hbar \frac{d\mathbf{k}}{dt} \qquad [\overleftrightarrow{\mathbf{M}}]^{-1} = \frac{1}{\hbar^2}\nabla_k \nabla_k E = \frac{c^2}{\hbar\omega}\left(\overleftrightarrow{\mathbf{I}} - \frac{1}{c^2}\mathbf{V}\mathbf{V}\right) \tag{7}$$

Let us now define the phase functions for a homogeneous and inhomogeneous waves that have for expression:

$$\varphi(\mathbf{r}, t) = \omega t - \mathbf{k} \cdot \mathbf{r} \qquad \varphi(\mathbf{r}, t) = \int_{t_0}^{t} \omega(\mathbf{r}, t')dt' - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{k}(\mathbf{r}', t_0) \cdot d\mathbf{r}' \tag{8}$$

Now, if an inhomogeneous Fourier mode $e^{j\varphi}$, with the phase given by the preceding equation, is a solution of a Klein-Gordon equation, it implies the following equations:

$$\frac{1}{c^2}\frac{\partial \omega}{\partial t} + \nabla \cdot \mathbf{k} = 0 \qquad \frac{\omega^2}{c^2} = \mathbf{k}^2 + k_0^2 \tag{9}$$

where the quantity k_0 in the above equation is a constant. Both homogeneous and inhomogeneous Fourier modes have the same dispersion law given by the second equation 9 with the difference that $\omega(\mathbf{r},t)$ and $\mathbf{k}(\mathbf{r},t)$ are functions of the Eulerian coordinates $\mathbf{r},t$.

In classical physics, particles and waves seem to exclude one another since a particle requires only one parameter, namely the position vector $\mathbf{R}(t)$ to describe a point particle while a wave requires two parameters the amplitude and the phase of the wave. Let us define the two group velocities:

$$\mathbf{V}_1(t)=\frac{d\mathbf{R}}{dt} \qquad \mathbf{V}_2(\omega)=\frac{\partial\omega}{\partial\mathbf{k}}=c^2\frac{\mathbf{k}}{\omega} \tag{10}$$

A priori, we have $\mathbf{V}_1\neq\mathbf{V}_2$, however, we can impose a constraint to the motion of the point particle by using the well-known formula of ray theory to link the two concepts, namely $\mathbf{V}_1(t)=\mathbf{V}_2(\omega)$ that implies to write the definitions $\omega[\mathbf{R}(t),t]=\omega(t)$ and $\mathbf{k}[(\mathbf{R}(t),t]=\mathbf{k}(t)$ are functions of the path coordinates $\mathbf{R},t$. A massive particle as a wave packet is composed of standing waves which satisfy in vacuum the above dispersion law for inhomogeneous Fourier modes. We have shown in our papers (25-28) that it is the above constraint between matter and wave which explains the De Broglie quantization relation $m_0\gamma\mathbf{V}=\hbar\mathbf{k}$ of matter.

To obtain the second quantization relation, it suffices to calculate the variation of the kinetic energies for the two velocities:

$$d\Big(\frac{\mathbf{V}_2^2}{2}\Big)=c^2(1-\frac{c^2k^2}{\omega^2})\frac{d\omega}{\omega}=\frac{c^2}{\gamma_2^2}\frac{d\omega}{\omega} \tag{11}$$

By definition, we have:

$$\frac{d\gamma_1}{dt}=\frac{\gamma_1^3}{c^2}\frac{d}{dt}\Big(\frac{\mathbf{V}_1^2}{2}\Big) \quad\Rightarrow\quad d\Big(\frac{\mathbf{V}_1^2}{2}\Big)=\frac{1}{\gamma_1^3}d(\gamma_1c^2) \tag{12}$$

The constraint $\mathbf{V}_1=\mathbf{V}_2$ implies a second relation:

$$\frac{dE}{E}=\frac{d\omega}{\omega} \tag{13}$$

We obtain the differential equation $d[ln(E/\omega)]=0$ which implies the second quantization relation $E(t)=m_0\gamma c^2=\hbar\omega(t)$. The energy $E(\omega)$ can be

related to the relativistic kinetic energy with the relation $\Delta E = E[\omega(t)] - E[\omega_0(t_0)] = E_K = m_0c^2(\gamma - 1)$ for the initial condition $\mathbf{V}(t_0) = 0$. The quantization of velocity during the acceleration process is confirmed by Jennison's experiments (29).

Therefore, the Planck $m_0\gamma c^2 = \hbar\omega$ and De Broglie $m_0\gamma\mathbf{V} = \hbar\mathbf{k}$ quantization relations are the result of a wave effect if we assume that a wave packet moving in a waving medium is built with inhomogeneous standing waves in vacuum as explained in several papers (30,31) and in our book (1,p.287). This point of view is shared by the authors of the book (32,p.310) when they discuss the nature of matter waves associated to a material particle. The similarity of quantization between classical scalar waves and quantum mechanical waves suggests a new interpretation of the quantization effect. Therefore, both quantization and wave-particle duality are described by the Planck and De Broglie relations:

$$E(t) = m_0\gamma(t)c^2 = \hbar\omega(t) \qquad \mathbf{P}(t) = m_0\gamma(t)\mathbf{V}(t) = \hbar\mathbf{k}(t) \qquad (14)$$

It is important to note that all the quantities γ, ω, k are function of time, a point which is never explicitly stated in books dealing with quantum mechanics.

These relations associate energy and momentum of a point particle with the angular frequency and wave vector of a plane wave. The Planck's constant $\hbar$ is merely a conversion factor between energy and frequency units. As explained in the reference (1,p.287) the quantization of the energy and momentum of the particle is often viewed as some kind of discontinuity in the physical behavior of the particle. In fact, the quantization results from a wave effect because the waves constituting the medium and the wave packet must deform themselves in a quantized manner in order to preserve their oscillatory behavior. In the reference (33), the author imagines that vacuum is a periodic medium as in solid state physics making our analogy a pertinent concept.

At the microscopique scale, matter and radiation can display particle or wave behavior. The question is how something can behave as a particle and as a wave at the same time. The answer given to that question by quantum mechanics is that the experimental set up determine when the experiment will display the particle or wave behavior. Therefore, any measurement gives either one property or the other but never both of them at the same time.

However, the ray condition law $\mathbf{V}_1 = \mathbf{V}_2$ and the analogy with the concept of effective mass in solid state physics contradict this statement. The Planck and De Broglie relations are derived from a common mathematical approach that ties the two aspects at the same time t. Therefore, there is no wave-particle duality, in other words, we have both aspects at the same time as proved by two experiments (34,35).

The experiment performed by Foster et al. (34) concerns the study of the wave particle duality in a way that has not been achieved in a single experiment before. The authors succeeded to detect light as both particle and wave at the same time, correlating the measured field amplitude of a wave property with the particle detection by photoelectric count. So light is observed directly in the experiment to act as a particle and a wave simultaneously. More recently in 2015, a research team at EPFL (35) has carried out an experiment showing that light can behave as both a wave and a stream of particles. A laser light propagating in a metallic nanowire produces by interference a standing wave which can radiates around the nanowire interacting with a stream of electrons fired along the nanowire to visualize the wave nature of light. At the same time, the electrons passing close to the confined light see their speed altered, moving faster or slower in a quantized manner. This change of speed appears as an exchange of photons showing the particule nature of light.

The quantities $\mathbf{k}$ and ω are often presented in the literature as parameters that we can adjust freely with the experiments. In fact, these quantities are function of time which defines the principal mode associated to a wave packet representing a material particle moving in a waving medium. The wave nature of matter was proved by the discovery of matter waves in the experiment done by Davisson and Germer in 1927 when they fired slow moving electrons at a crystalline nickel target.

3. THE HARMONIC OSCILLATOR

The harmonic oscillator is one of the most important model systems since various physical situations can be explained with this system particularly in quantum mechanics. There are in the literature many papers published on this subject but none of the authors of these papers are able to demonstrate the quantization effect from first principles in a simple manner. The most general differential equation to describe an harmonic oscillator has for expression:

$$m(t)\,\frac{d^2\mathbf{R}}{dt^2} + \alpha(t)\,\frac{d\mathbf{R}}{dt} + k(t)\,\mathbf{R} = \mathbf{F} \qquad (15)$$

where the mass $m(t)$ of the particle and the spring constant $k(t)$ are all positive functions of time. The function $\alpha(t)$ can negative if we have a sink term for absorption of energy or positive in the case of a source term for emission of energy by the system.

The force function $\mathbf{F}$ can be partitioned between internal and external forces but can be also integrated in an implicit manner as suggested by Gitterman (36) in the definition of the three parameter functions in order to solve an homogeneous differential equation.

In two interesting papers (37,38) published in 2003 and 2019, Boyer presented a new approach concerning the blackbody problem where the classical harmonic oscillator provides a simple thermodynamic system with an adiabatic invariant E_M/ω for the mechanical system. Moreover, his analysis emphasizes the intimate connections between blackbody radiation and zero-point radiation.

3.1. Case of the classical harmonic oscillator

Let us first review the classical harmonic oscillator of mass m_0 where the spring coefficient k_0 is constant. The equation of motion for the displacement of the mass m_0 has for expression:

$$m_0\,\frac{d^2\mathbf{R}}{dt^2} = \mathbf{F}_i \quad \Rightarrow \quad \frac{d^2\mathbf{R}}{dt^2} + \omega^2\mathbf{R} = 0 \qquad (16)$$

where the internal force $\mathbf{F}_i$ derives from a potential $E_P[\mathbf{R}(t)] = k_0\mathbf{R}^2/2$. The solution of the preceding equation is $\mathbf{R}(t) = \mathbf{R}_0\cos(\omega t + \theta)$ where the angular frequency ω is given by the definition $m_0\omega^2 = k_0$.

Since the quantity k_0 is constant, the potential does not depend explicitly on time, therefore, the system is closed and the mechanical energy E_M is also constant:

$$E_M = \frac{1}{2} m_0 \mathbf{V}^2 + \frac{1}{2} k_0 \mathbf{R}^2 = Ct \tag{17}$$

It is well known that we can classify systems in closed or open systems depending on whether the mechanical energy is conserved or not. Therefore, one must make the distinction between an internal force which always follows the conservation law of energy and an external force which is not conservative and can be considered as a source term for the system.

We now assume that the interaction of an internal force with the harmonic oscillator can be implicitly taken into account by making the spring constant a function of time with a potential function which now depends explicitly on time $E_P[\mathbf{R}(t), t] = k(t)\mathbf{R}^2(t)/2$. Therefore, the particular derivative of this function becomes:

$$\frac{dE_P}{dt} = \frac{\partial E_P}{\partial t} + \mathbf{V} \cdot \nabla E_P = \frac{1}{k} \frac{dk}{dt} E_P + k\, \mathbf{R} \cdot \mathbf{V} \tag{18}$$

knowing that:

$$\frac{\partial E_P}{\partial t} = \frac{1}{2} \frac{dk}{dt} \mathbf{R}^2(t) = \frac{1}{k} \frac{dk}{dt} E_P \tag{19}$$

We have now:

$$\frac{d^2\mathbf{R}}{dt^2} + \omega^2(t)\mathbf{R} = 0 \quad with \quad \omega^2(t) = \frac{k(t)}{m_0} \tag{20}$$

If we multiply the preceding equation by $\mathbf{V}$, we get:

$$\frac{dE_K}{dt} = -k\, \mathbf{R} \cdot \mathbf{V} \tag{21}$$

If we add both equations 18 and 21, we obtain the relation:

$$\frac{dE_M}{dt} = \frac{1}{k} \frac{dk}{dt} E_P = \frac{2}{\omega} \frac{d\omega}{dt} E_P \tag{22}$$

with the new definition

$$E_M(t) = \frac{1}{2} m_0 \mathbf{V}^2 + \frac{1}{2} k(t) \mathbf{R}^2 \tag{23}$$

Since the spring function $k(t)$ changes with time, the system is not closed and therefore the mechanical energy is no longer a constant and will increase for $dk/dt > 0$.

Ehrenfest (39) noted that certain quantities remained constant during slow change of a system. This lead him to introduce his adiabatic hypothesis suggesting that these quantities were the elements to take the discrete values of quantum mechanics. Now, if the change in the spring constant $k(t)$ is very slow, we may replace the potential energy E_P by its time average $E_P(T)$ over a period as suggested by Ehrenfest:

$$E_P(T) = \frac{1}{T}\int_0^T E_P[\mathbf{R}(t'), t']dt' = \frac{E_M}{2} \tag{24}$$

We note that the preceding equality is strictly verified when k is constant. If we substitute the above relation into equation 22, we get:

$$\frac{dE_M}{dt} = \frac{1}{\omega}\frac{d\omega}{dt}E_M \quad \Rightarrow \quad d[ln(E_M/\omega)] = 0 \tag{25}$$

The quantity E_M/ω stays constant during the change of k, it follows the relation $E_M = \hbar\omega$ where the Planck's constant is independent of ω.

We remark that the oscillatory motion is frequency modulated since $\omega(t)$ is a function of time. We note also that the quantization effect applied to both the kinetic and potential energies while it applied only to the kinetic energy in the Planck and De Broglie relations. It is instructive to use another approach to solve the differential equation:

$$\frac{dE}{dt} - \alpha(t)\,E = 0 \quad with \quad \alpha(t) = \frac{1}{\omega}\frac{d\omega}{dt} \tag{26}$$

We know that the solution of the above equation can be expressed in terms of exponential functions

$$E(t) = E_0\, e^{y(t)} \quad with \quad y(t) = \int_0^t \alpha(t')dt' \tag{27}$$

with the initial condition $E_0 = E(0)$. Knowing that

$$y(t) = ln\frac{E}{E_0} = \int_0^t \alpha(t')dt' = \int_0^t \frac{d}{dt'}ln[\omega(t')]dt' = ln\frac{\omega}{\omega_0} \tag{28}$$

with the initial condition $\omega_0 = \omega(0)$. Therefore, the mechanical energy depends on a linear manner upon the frequency of the wave since we have:

$$E(t) = \frac{E_0}{\omega_0}\omega(t) \tag{29}$$

where the Planck's constant $\hbar$ is indeed a dimensional constant relating energy and frequency units $E_0 = \hbar\,\omega_0$. The above analysis still applies in the case where the potential energy is of the form $E_P[\mathbf{R}(t), t] = -k(t)/R(t)$.

3.2.Frequency modulation of a harmonic oscillator

Let us assume that the displacement $\mathbf{R}(t)$ of a frequency modulated harmonic oscillator is given by the expression:

$$\mathbf{R}(t) = \mathbf{R}_0 \cos[\varphi(t)] \quad with \quad \varphi(t) = \int_0^t \omega^n(t')dt' \tag{30}$$

If we derive the function $\mathbf{R}$ two times with respect to time, we get the equation of motion:

$$m_0 \frac{d^2\mathbf{R}}{dt^2} = -m_0\omega^{2n}\mathbf{R} + m_0 \frac{n}{\omega}\frac{d\omega}{dt}\mathbf{V} = \mathbf{F}_i + \mathbf{F}_e \tag{31}$$

Therefore, the function $\mathbf{R}$ is an exact solution of the differential equation:

$$\frac{d^2\mathbf{R}}{dt^2} - \frac{n}{\omega}\frac{d\omega}{dt}\frac{d\mathbf{R}}{dt} + \omega^{2n}\mathbf{R} = 0 \tag{32}$$

If we multiply the preceding equation by $\mathbf{V}$, we get:

$$\frac{dE_K}{dt} = \frac{1}{k}\frac{dk}{dt}E_K - k\,\mathbf{R}\cdot\mathbf{V} \tag{33}$$

knowing that the kinetic energy has for expression $E_K = m_0\mathbf{V}^2/2$ and where the spring variable $k(t)$ has now for definition:

$$k(t) = m_0\omega^{2n}(t) \quad \Rightarrow \quad \frac{2n}{\omega}\frac{d\omega}{dt} = \frac{1}{k}\frac{dk}{dt} \tag{34}$$

Since $\omega(t)$ depends on time, the above equation implies the existence of a force $\mathbf{F}_e$ which is a sink term if the condition $d\omega/dt > 0$ is satisfied or a source term for $d\omega/dt < 0$ for the harmonic oscillator.

The variation of the potential energy calculated from the function $E_P = k(t)\mathbf{R}^2/2$ has for expression

$$\frac{dE_P}{dt} = \frac{1}{k}\frac{dk}{dt}E_P + k\,\mathbf{R}\cdot\mathbf{V} \tag{35}$$

If we add the equations 33 and 35, we obtain:

$$\frac{dE_M}{dt} = \frac{2n}{\omega}\frac{d\omega}{dt}E_M \quad \Rightarrow \quad d[ln(E_M/\omega^{2n})] = 0 \tag{36}$$

For $n = 1/2$, we get the classical quantization law for energy $E_M = \hbar\omega$ while for $n = 1$, we get $E_M = Ct\,\omega^2$ which is the quantization law in solid state physics for an apparent free particle of mass m_0 since we have:

$$E[\omega(\mathbf{k})] - m_0c^2 = \frac{\hbar^2\omega^2}{2m_0c^2} \tag{37}$$

The mechanical energy can increase or decrease in a slow manner with an oscillatory motion superposed to the slow motion since we have the equations:

$$E_K(t) = E_M(t)\sin^2[\varphi(t)] = \frac{1}{2}E_M(t) - \frac{1}{2}E_M(t)\cos[2\varphi(t)] \tag{38}$$

$$E_P(t) = E_M(t)\cos^2[\varphi(t)] = \frac{1}{2}E_M(t) + \frac{1}{2}E_M(t)\cos[2\varphi(t)] \tag{39}$$

knowing that $E_M(t) = k(t)R_0^2/2$ where the function $k(t)$ is a positive quantity by definition. Our calculation is classical and exact and show how we can obtain the usual quantization law for the mechanical energy instead of the kinetic energy alone as demonstrated above. This new approach gives an insight in the quantization effect which arises when there is an implicit force that depends on a time dependent angular frequency.

3.3. Case of the time-dependent mass harmonic oscillator

The relativistic formulation of the Planck-De Broglie relations suggests assuming that the mass of the particle $m(t)$ is a function of time that will allow to recover a quantization relation. In that case, we can define the momentum and the kinetic energy of the material particle by the relations:

$$\mathbf{P}(t) = m(t)\,\mathbf{V} \qquad E_K = \frac{1}{2}\,m(t)\,\mathbf{V}^2 \tag{40}$$

The equation of motion of the particle is:

$$\frac{d\mathbf{P}}{dt} = -k(t)\,\mathbf{R} = \mathbf{F}_i \quad \Rightarrow \quad \frac{d^2\mathbf{R}}{dt^2} + \frac{d}{dt}\,ln(m)\,\frac{d\mathbf{R}}{dt} + \omega^2\mathbf{R} = 0 \tag{41}$$

where $\omega(t)$ has now for definition:

$$\omega^2(t) = \frac{k(t)}{m(t)} \quad \Rightarrow \quad \frac{2}{\omega}\,\frac{d\omega}{dt} = \frac{1}{k}\,\frac{dk}{dt} - \frac{1}{m}\,\frac{dm}{dt} \tag{42}$$

We multiply the equation of motion by the velocity $\mathbf{V}$, we get a first equation

$$\mathbf{V}\cdot\frac{d\mathbf{P}}{dt} = m\,\mathbf{V}\cdot\frac{d\mathbf{V}}{dt} + \frac{2}{m}\,\frac{dm}{dt}\,E_K = -k\,\mathbf{R}\cdot\mathbf{V} \tag{43}$$

and a second equation by deriving the kinetic energy definition:

$$\frac{dE_K}{dt} = m\,\mathbf{V}\cdot\frac{d\mathbf{V}}{dt} + \frac{1}{m}\,\frac{dm}{dt}\,E_K \tag{44}$$

Finally, the two preceding equations provide a first identity

$$\frac{dE_K}{dt} = -\frac{1}{m}\,\frac{dm}{dt}\,E_K - k\,\mathbf{R}\cdot\mathbf{V} \tag{45}$$

The second identity is obtained by the variation the potential energy already calculated in equation 35:

$$\frac{dE_P}{dt} = \frac{1}{k}\,\frac{dk}{dt}\,E_P + k\,\mathbf{R}\cdot\mathbf{V} \tag{46}$$

If we add the two above equations, we obtain:

$$\frac{dE_M}{dt} = \frac{1}{k}\frac{dk}{dt}E_P - \frac{1}{m}\frac{dm}{dt}E_K \tag{47}$$

with a new definition for the mechanical energy

$$E_M(t) = \frac{1}{2}m(t)\,\mathbf{V}^2 + \frac{1}{2}k(t)\mathbf{R}^2 \tag{48}$$

If we use the Ehrenfest condition $E_K \approx E_M/2$ and $E_P \approx E_M/2$ in equation 47, then this equation becomes:

$$\frac{dE_M}{dt} = \frac{1}{\omega}\frac{d\omega}{dt}E_M \quad \Rightarrow \quad d[ln(E_M/\omega)] = 0 \tag{49}$$

It follows that the mechanical energy is quantized with the classical law $E_M = \hbar\omega$ if the mass $m(t)$ depends on time. We recall that the mass $m(t)$ is a positive quantity that can increase or decrease, therefore, the term $dm/dt\,\mathbf{V}$ in equation 41 is a dissipative term for $dm/dt > 0$ and the harmonic oscillator is a damped oscillator in that case. The preceding formulations of the harmonic oscillator can be used to explain the quantum Hall effect (40,41), since the allowed energy levels for a free two-dimensional electron system moving in a magnetic field are identical to a fictitious one-dimensional harmonic oscillator.

4. THE PHOTOELECTRIC EFFECT

The photoelectric effect is an individual collision process between a bound electron in the atom and an incoming photon particle. This effect is a pillar of the foundation of quantum mechanics since it was the first experiment to show clearly the wave-particle duality of light. Indeed, several features of the photoelectric effect cannot be explained by the classical wave theory of light, namely:

- It is expected that the kinetic energy of a photoelectron should increase with the intensity of incident light. However, it is an experimental fact that the kinetic energy of the photoelectrons does not depend on intensity.

- Classical wave theory predicts that photoemission should occur at all frequencies of incident light. On the contrary, we observe that there exists a

frequency, specific to each metal, such that light of shorter frequency cannot liberate electrons from the metallic surface.

- Classically, an electron would not be emitted from the surface of the metal until the incident light transferred enough of its energy to the electron. This would require an important delay between the time the light is absorbed by the electron and the time the electron is ejected. However, the emission of electrons occurs very shortly after the arrival of the radiation in a laps of time smaller than 10^{-9} s.

Therefore, the electromagnetic wave picture of light completely fails to explain the features of the photoelectric effect while the granular nature of light can explain why light can be absorbed in discrete amounts. This can explain the reason why there are so little papers (42-45) published in the literature concerning a wave approach of the photoelectric effect since 1905 when Einstein write his famous equation:

$$\frac{1}{2} m_0 \mathbf{V}^2 + \hbar\omega_s = \hbar\omega_p \tag{50}$$

where the first term is the kinetic energy of the ejected electron, the second term $\hbar\omega_s$ is the energy threshold required for an electron to escape the surface of the metal and the third term $\hbar\omega_p$ is the energy of the incident photon fired towards the metallic plate. Millikan reported his experimental results in the Physical Review in March 1916. His data points fell on a perfectly straight line of slope h/e leaving no doubt concerning the validity of Einstein's equation of the photoelectric effect.

This equation played an enormous part in the development of modern quantum mechanics. But in spite of its generality and of the many successful applications that have been made since, his equation based on the concept of photon is completely at variance with the most fundamental concepts of the classical electromagnetic theory of radiation. Indeed, how a photon as a point particle without mass and charge can interfere or even be absorbed by an electron, also a point particle with mass and charge.

However, in the present theory, the incident light wave perturbs the oscillatory frequency of the harmonic oscillator which results in a frequency modulation between an initial state of the system at time $t = 0$ when light is fired towards the metallic plate:

$$E_K(0) + E_P(0) = \hbar\omega(0) \tag{51}$$

and the final state of the system at time $t = \Delta t$ where Δt is the time lag between the incident wave and the time when the electron is ejected from the metallic plate.

$$E_K(\Delta t) + E_P(\Delta t) = \hbar\omega(\Delta t) \tag{52}$$

Finally, we can substrat the preceding equations, to get the identities:

$$E_K(\Delta t) - E_K(0) = \frac{1}{2} m_0 \mathbf{V}^2 \tag{53}$$

$$E_P(\Delta t) - E_P(0) = \hbar\omega_s \tag{54}$$

$$\hbar\omega(\Delta t) - \hbar\omega(0) = \hbar\omega_p \tag{55}$$

We recover the Einstein's relation but with a different meaning. Indeed, the real question is how a wave can interact with a material particle. The resolution of this problem in terms of electromagnetic wave motion interacting with the motion of a material particle can be solved if the material particle is a wave packet where both the light pulse and the particule wave packet move in a waving medium. This fact has been proven in several papers (30,31) where it has been shown that the electromagnetic field and the Maxwell's equations result from the space-time deformation of scalar standing waves in vacuum explaining for the first time why vacuum can sustain transverse waves.

Visible light with a spectrum $4.3\,10^{14}$ Hz for red light to $7.5\,10^{14}$ Hz for violet light is used to produce the photoelectric effect. For the yellow line of sodium vapor, the frequency is $F = 5.1\,10^{14}$ Hz, the period is $T = 1.96\,10^{-15}$ s with a time coherence about $T_c = 2\,10^{-12}$ s. The spring constant k in a harmonic oscillator is about $k \approx 10^3$ N-m which implies a frequency $F \approx 5.3\,10^{15}$ Hz, one order greater than the frequency of the incident wave. The time lag Δt between the incident wave and the time when the electron is ejected is determined by the frequency modulation of the harmonic oscillator since we must have $2\,10^{-15} < \Delta t < 2\,10^{-12}$ s.

The nineteenth century physicists examined the photoelectric effect using conventional lamps. Therefore, a physicist of that time could not detect multiphoton electron emission. Today, with high intensity laser light and very fast electronics, we can detect multiphoton electron emission since

the rate of multiphoton emission is greater with lasers than with lamps of light. The theory of the multiphoton electron emission is still a field of research to day. We can quote several experiments (47-50) concerning the subject.

We can find in the literature, two papers titled "How a Photon is Created or Absorbed" (51,52), these papers deal with the emission or absorption of a photon by an atom which is often depicted as if it is an instantaneous phenomenon. However, this point of view is incorrect since there is a time dependence of the wave function in a quantum transition. Experimental measurements have been made of samples undergoing spectroscopic transitions. These experiments confirm that there are no instantaneous quantum jumps. Instead, the electric and magnetic fields undergo a smooth and continuous transition occurring over the emission or absorption time period confirming the wave point of view adopted to describe the photoelectric effect.

There are numerous experimental observations that confirm that photons are emitted or absorbed over a time period corresponding to the inverse bandwidth. Quoting from the above article:

"The first experimental measurements of bulk samples undergoing spectroscopic transitions were obtained from nuclear magnetic resonance observations of the transition nutation effect and spin echoes 5, 6 using coherent radiation produced by a single radio frequency oscillator. More recently, the analogous transition nutation effect 7, 8 and so called "photon echoes" 9, 10, 11 have been observed in molecular spectra using pulsed coherent laser radiation. These experiments confirm that there is no "quantum jumps" in the non stationary state"

The authors also conclude their paper by the statement:

"In summary, these methods of illustrating the dynamics of spectroscopic transitions clearly reveal the quantum mechanical origin of oscillating transition moments and the characteristic resonance between the system and the radiation during the creation or absorption of a photon. They provide statistical information on the trajectories of particles which correlate with classical descriptions."

Many experiments since the classic 1916 experiments of Millikan have shown that there is photoelectric current even for frequencies below the threshold, contrary to the explanation by Einstein, as shown by Richard

Keesing (53) who made a critical review of the Millikan's 1916 papers where his original experimental data show an asymptotic behavior of the photocurrent versus voltage with no clear threshold frequency.

We can also quote a recent conference where the authors (54) sum up the role played by the adiabatic hypothesis in the development of the old quantum theory that was formulated by Ehrenfest in a paper published in 1916.

5. CONCLUSION

In the present paper, we have proved that the Planck and De Broglie relations can be obtained from first principles if we use the ray theory which implies that there is no wave-particle duality provided that material particles have a finite size, that is to say, that material particles must be real wave packets. The quantization effect can be explained if we introduce the concept of scalar inhomogeneous waves in vacuum which can be the scalar field associated to the so called zero point energy. We demonstrated how the concept of quantization can be generalized to the case of the harmonic oscillator with an application to the photoelectric effect which can now be explained in a classical way as a wave effect without photon. This new approach is in agreement with several experiments (29,34,35,51,52) quoted in this paper proving the existence of particles with finite size and the non existence of the wave-particle duality as explained in textbooks.

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EXAMINING A SURFACE BASED ON A DEVELOPABLE RULED SURFACE

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Abstract

Developable ruled surfaces are a wide range of manufactured surfaces, since they can be unfolded or developed into a plane without tearing or stretching. In this paper, a surface is examined using a developable ruled surface. Our contribution is that we can express three developable ruled surfaces and their relationships with each other at any point on a surface. We give some results and theorems to present that every surface can be studied using developable ruled surfaces.

keywords: Developable ruled surface; lines of curvature; parameter curve; tangent indicatrix.

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1 Introduction

A surface including one parameter family of straight lines is called as a ruled surface. These surfaces are widely used in many areas of science such as computer-aided geometric design (CAGD), differential geometry, robotic geometry and kinematics.

Developable ruled surfaces are one of the most important subject in both theory and practice. Since, these surfaces can be unfolded (developed) into a plane without stretching or tearing, developable ruled surfaces are relevant to the sheet- metal and plate-metal based industries [1], [2], ship hulls [3] and clothing [4], aircraft skins [5], architectural structures [6] and art [7]. A developable ruled surface can be a plane, a conical surface, a cylindrical surface or tangential surface of a curve [8]. Differential geometry provides a detailed mathematical and geometrical information of developable ruled surfaces before CAGD. The literature on developable surfaces is quite widespread. Chu and Frey [9], [10] studied on the properties and characteristics of developable surfaces. Also, some researchers are focused on the construction and exact representation of these surfaces [11], [12], [13], [14], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25].

A tangent vector at each point of a curve on a surface is in a principal direction, and this tangent vector is called as a line of curvature. If normal curvatures are equal at the point of a curve on a surface, this point is called as an umbilical point. The principal directions are indeterminate at the umbilical points. Therefore, there are two principal directions that are orthogonal to each other at each non-umbilical point. Thus, orthogonal lines of curvature form an orthogonal net of lines [26]. Che et al. [27] presented the lines of curvature and their differential geometry properties.

In this paper, we look into a surface using a developable ruled surface. Also, we give three developable ruled surfaces and their relationships with each other at any point on a surface. Our contribution is that every regular surface can be examined using developable ruled surfaces.

2 Preliminaries

Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a spatial curve in $\mathbb{E}^3$. The curve α is a unit speed curve if the condition $\|\alpha'(s)\| = 1$ is satisfied, where s is the arc length parameter of this curve. Serret-Frenet frame of this curve is,

$$\alpha'(s) = t \qquad \frac{t'}{\|t'\|} = N \qquad t \times N = B \tag{2.1}$$

where the vectors $\mathbf{t}, \mathbf{N}$ and $\mathbf{B}$ are the tangent unit vector, principal unit normal vector and binormal unit normal vector. The curvature and the torsion of the curve $\alpha(s)$ at the point P are $\kappa = \kappa(s) = \|\mathbf{t}'(s)\|$ and $\tau = \tau(s) = \|\mathbf{B}'(s)\|$, respectively. The derivative formulas of the Serret-Frenet frame are defined as

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{2.2}$$

The Frenet vector and the unit Frenet vector of $\alpha(s)$ are written as

$$\begin{aligned} \mathbf{f} &= \tau\mathbf{t} + \kappa\mathbf{B} \\ \mathbf{w} &= \frac{\mathbf{f}}{\|\mathbf{f}\|} = \frac{\tau\mathbf{t}+\kappa\mathbf{B}}{\sqrt{\tau^2+\kappa^2}} \end{aligned} \tag{2.3}$$

A regular surface is given as $\mathbf{r}(u, v)$ and its parameter curves are $u = fxd.$ and $v = fxd$ with arc length parameter s_1 and s_2. Let's take a point P on the surface $\mathbf{r}(u, v)$, and take a curve $\alpha(s)$ that passes through the point P with arc length parameter s. Since, the curve $\alpha(s)$ lies on the surface $\mathbf{r}(u, v)$, it has another frame that it is called as a Darboux frame $\{\mathbf{t}, \mathbf{g}, \mathbf{n}\}$ of the curve $\alpha(s)$ at P, in which the vectors $\mathbf{t}, \mathbf{g}$, and $\mathbf{n}$ are the unit tangent vector, the unit vector defined as $\mathbf{g} = \mathbf{n} \times \mathbf{t}$, and the unit normal vectors, respectively. The derivative formulas of Darboux frame are as follows:

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{g}' \\ \mathbf{n}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{g} \\ \mathbf{n} \end{bmatrix}, \tag{2.4}$$

where, κ_g is the geodesic curvature, κ_n is the normal curvature, and τ_g is the geodesic torsion of the curve $\alpha(s)$. The Darboux vector and the unit Darboux vector of this curve are given as follows [28]:

$$\mathbf{d} = \tau_g \mathbf{t} + \kappa_n \mathbf{g} + \kappa_g \mathbf{n}, \qquad \mathbf{c} = \frac{\mathbf{d}}{\|\mathbf{d}\|} = \frac{\tau_g \mathbf{t} + \kappa_n \mathbf{g} + \kappa_g \mathbf{n}}{\|\tau_g \mathbf{t} + \kappa_n \mathbf{g} + \kappa_g \mathbf{n}\|} \tag{2.5}$$

Next, the first and second fundamental quantities, the Gauss and the mean curvatures of the regular surface $\mathbf{r}(u, v)$ can be defined as [29]:

$$\begin{aligned} E &= \mathbf{r_u^2}, \qquad F = \mathbf{r_u r_v}, \qquad G = \mathbf{r_v^2} \\ L &= \mathbf{n.r_{uu}}, \qquad M = \mathbf{n.r_{uv}}, \qquad N = \mathbf{n.r_{vv}} \end{aligned} \tag{2.6}$$

and

$$K = \frac{LN - M^2}{EG - F^2} \qquad \text{and} \qquad 2H = \frac{EN - 2FN + GL}{EG - F^2}. \tag{2.7}$$

On the other hand, the parameter curves $u = fxd.$ and $v = fxd$ of the surface $\mathbf{r}(u, v)$ have their own Darboux frames as $\{\mathbf{t_1}, \mathbf{g_1}, \mathbf{n_1}\}$ and $\{\mathbf{t_2}, \mathbf{g_2}, \mathbf{n_2}\}$, respectively (see Figure 1).
The derivative formulas of these Darboux frames are:

$$\begin{bmatrix} \mathbf{t_1'} \\ \mathbf{g_1'} \\ \mathbf{n'} \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g^{(1)} & \kappa_n^{(1)} \\ -\kappa_g^{(1)} & 0 & \tau_g^{(1)} \\ -\kappa_n^{(1)} & -\tau_g^{(1)} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t_1} \\ \mathbf{g_1} \\ \mathbf{n} \end{bmatrix} \tag{2.8}$$

and

$$\begin{bmatrix} \mathbf{t_2'} \\ \mathbf{g_2'} \\ \mathbf{n'} \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g^{(2)} & \kappa_n^{(2)} \\ -\kappa_g^{(2)} & 0 & \tau_g^{(2)} \\ -\kappa_n^{(2)} & -\tau_g^{(2)} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t_2} \\ \mathbf{g_2} \\ \mathbf{n} \end{bmatrix}. \tag{2.9}$$

The unit tangent vectors of the parameter curves $(v = fxd.)$ and $(u = fxd.)$ can be expressed as follows:

$$\begin{aligned} \mathbf{t_1} &= \frac{\mathbf{r_u(u,v)}}{\|\mathbf{r_u(u,v)}\|} = \frac{\mathbf{r_u}}{\sqrt{EG-F^2}}, \\ \mathbf{t_2} &= \frac{\mathbf{r_v(u,v)}}{\|\mathbf{r_v(u,v)}\|} = \frac{\mathbf{r_v}}{\sqrt{EG-F^2}}. \end{aligned} \tag{2.10}$$

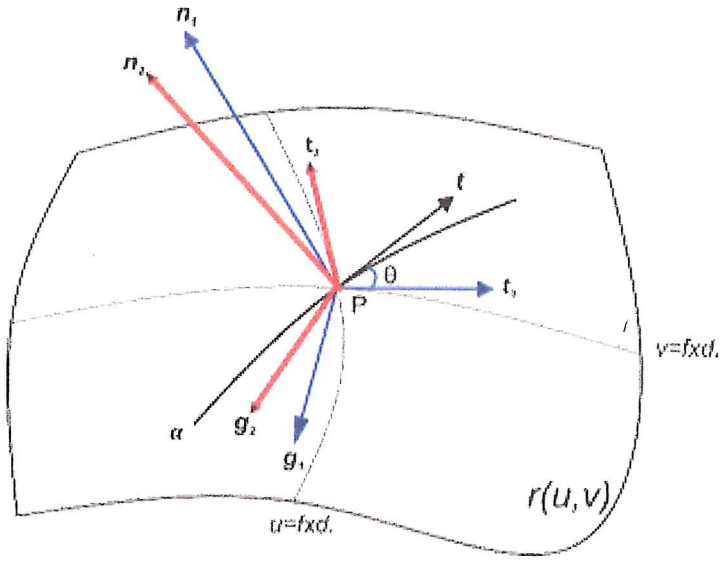

Figure 1: The relationship between the Darboux frames of the parameter curves $u = fxd.$ and $v = fxd.$ of the surface $\mathbf{r}(u, v)$

where $\mathbf{r_u}$ and $\mathbf{r_v}$ are the partial derivatives of $\mathbf{r(u, v)}$ according to parameters u and v [30].

Let's suppose that the parameter curves $\mathbf{r_u(u, v)}$ and $\mathbf{r_v(u, v)}$ are lines of curvature and the angle between the unit vectors $\mathbf{t}$ and $\mathbf{t_1}$ is θ shown in Figure (2). Then, the relationship between Darboux frame vectors of the curve α and the unit tangent vectors of the parameter curves is given with below equations.

$$\mathbf{t} = \mathbf{t_1}\cos\theta + \mathbf{t_2}\sin\theta, \quad \mathbf{g} = \mathbf{t_1}\sin\theta - \mathbf{t_2}\cos\theta, \quad \mathbf{n} = \mathbf{t_1} \times \mathbf{t_2} \tag{2.11}$$

Since, the parameter curves $\mathbf{r_u(u, v)}$ and $\mathbf{r_v(u, v)}$ are lines of curvature, unit tangent vectors of these curves are written as

$$\mathbf{t_1} = \frac{r_u}{\sqrt{E}} \qquad \text{and} \qquad \mathbf{t_2} = \frac{r_v}{\sqrt{G}}. \tag{2.12}$$

2.1 Developable Surface

A ruled surface is a surface which is generated by one parameter family of straight lines and given as;

$$\mathbf{X}(u,v) = \mathbf{a}(v) + u\mathbf{b}(v) \tag{2.13}$$

where, $\alpha(v)$ is the base curve or the directrix and $\mathbf{l}(u) = \mathbf{a}(v_0) + u\mathbf{b}(v_0)$ is a line and called as the generator or the ruling of the ruled surface $\mathbf{X}(u,v)$ for a constant value $v = v_0$.

A ruled surface is called as a developable ruled surface if the tangent plane is constant at all points along a given ruling [29]. A developable ruled surface can be a plane, a conical surface, a cylindrical surface or tangential developable surface (see Figure2).

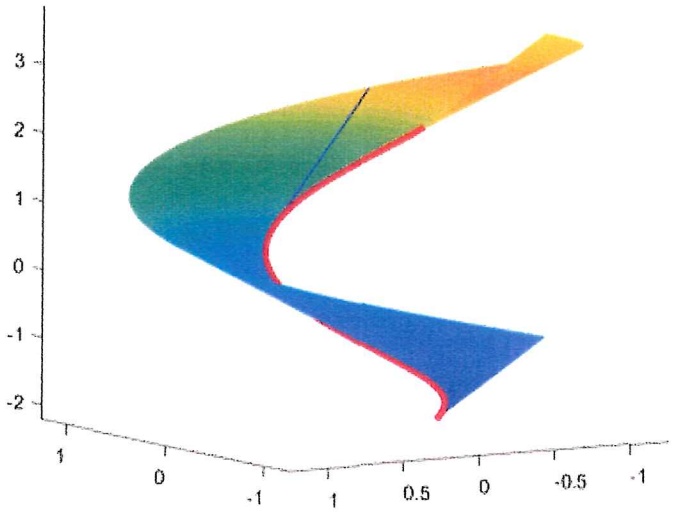

Figure 2: Ruling (blue) and directrix curve (red) of a tangential developable surface.

3 Analyzing A Surface According to The Developable Ruled Surface Related to This Surface

Developable ruled surfaces which take the parameter curves ($v = fxd.$) and ($u = fxd.$), and the curve $\alpha(s)$ as the directrix curves (see Figure 3) at the point P on the surface $\mathbf{r}(u, v)$ are defined using the equation (11) as follows (see Figure 4):

$$\mathbf{X_1}(s_1, u_1) = \alpha_1(s_1) + u_1\mathbf{t_1}(s_1) = \alpha_1(s_1) + u_1\frac{\mathbf{r_u}(s_1)}{\sqrt{E}}, \tag{3.1}$$

$$\mathbf{X_2}(s_2, u_2) = \alpha_2(s_2) + u_2\mathbf{t_2}(s_2) = \alpha_2(s_2) + u_2\frac{\mathbf{r_v}(s_2)}{\sqrt{G}}, \tag{3.2}$$

$$\mathbf{X}(s, u) = \alpha(s) + u\mathbf{t}(s) = \alpha(s) + (\mathbf{t_1}\cos\theta + \mathbf{t_2}\sin\theta)(s). \tag{3.3}$$

Additionally, the rulings of these developable ruled surfaces can be written by below equations:

$$\begin{aligned} \mathbf{l_1} &= \alpha_1(s_{10}) + u_1\mathbf{t_1}(s_{10}), \\ \mathbf{l_2} &= \alpha_2(s_{20}) + u_2\mathbf{t_2}(s_{20}), \\ \mathbf{l} &= \alpha(s_0) + u\mathbf{t}(s_0). \end{aligned} \tag{3.4}$$

3.1 The Arc Length Parameters and The Tangent Indicatrices of The Directrix Curves of The Developable Ruled Surfaces

It is expressed that the parameter curves are lines of curvatures. Hence, we can write the arc length parameters of parameter curves ($v = fxd.$) and

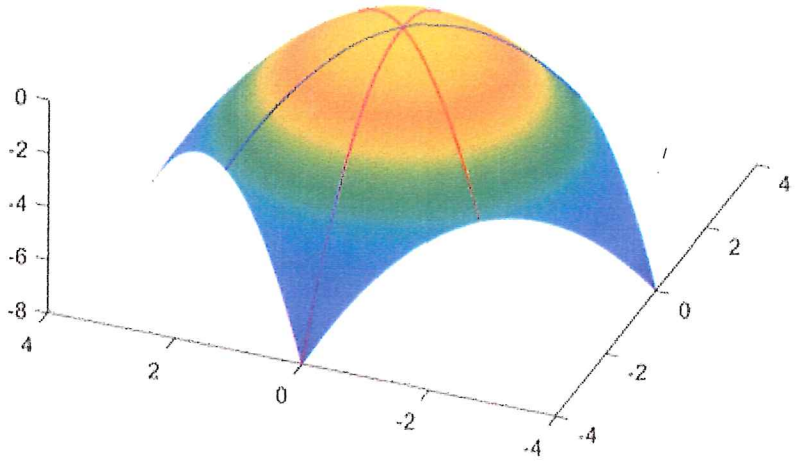

Figure 3: Parameter curves ($v = fxd.$) (red) and ($u = fxd.$)(blue), and the curve $\alpha(s)$(magenta) as the directrix curves.

($u = fxd.$), in terms of the coefficients of the first fundamental form of the surface $\mathbf{r}(u, v)$ respectively as:

$$ds_1^2 = Edu^2 \qquad \text{and} \qquad ds_2^2 = Gdv^2 \tag{3.5}$$

The relationship between the arc length parameters of parameter curves ($v = fxd.$) and ($u = fxd.$), and curve $\alpha(s)$ can be given with following equation.

$$\begin{aligned} \frac{ds_1}{ds} &= \sqrt{E}\frac{du}{ds} = \cos\theta, \\ \frac{ds_2}{ds} &= \sqrt{G}\frac{dv}{ds} = \sin\theta, \\ ds^2 &= ds_1^2 + ds_2^2. \end{aligned} \tag{3.6}$$

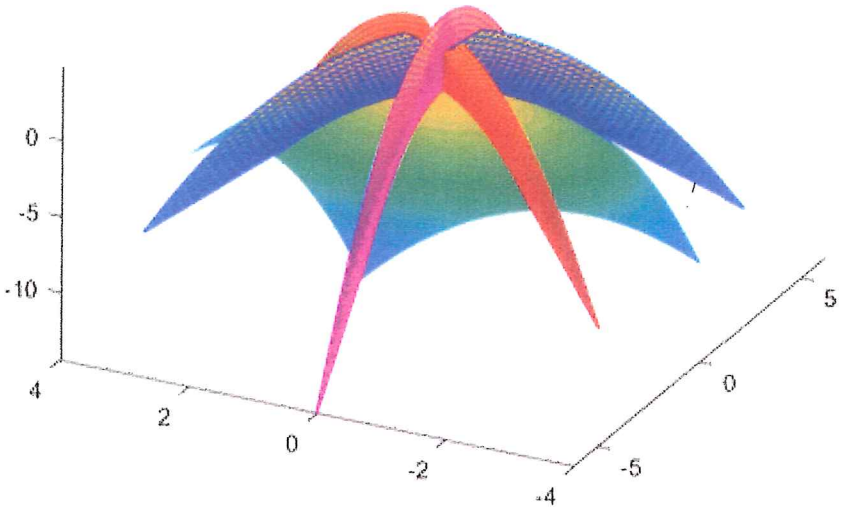

Figure 4: Developable ruled surfaces that the red one takes the parameter curve ($v = fxd.$) and the blue one takes the parameter curve ($u = fxd.$), and the magenta one takes the curve $\alpha(s)$ as the directrix curves at the same point on a surface.

Next, the tangent indicatrices of the parameter curves ($v = fxd.$) and ($u = fxd.$), and curve $\alpha(s)$ is defined as $\frac{ds_{1t}}{ds_1} = \|\mathbf{t_1'}\| = \kappa_1$ and $\frac{ds_{2t}}{ds_2} = \|\mathbf{t_2'}\| = \kappa_2$. Using the first and second equations of (18), following relations can be written.

$$\begin{aligned}
\frac{ds_{1t}}{ds_1} &= \frac{ds_{1t}}{ds} = \kappa_1 \cos\theta, \\
\frac{ds_{2t}}{ds_2} &= \frac{ds_{2t}}{ds} = \kappa_2 \sin\theta, \\
\frac{ds_{1t}}{ds_{2t}} &= \frac{\kappa_1 \cos\theta ds}{\kappa_2 \sin\theta ds} = \frac{\kappa_1}{\kappa_2} \cot\theta,
\end{aligned} \tag{3.7}$$

and

$$\frac{ds_{1t}}{ds_t} = \frac{ds_{1t}}{ds} = \frac{\kappa_1}{\kappa}\cos\theta,$$

$$\frac{ds_{2t}}{ds_2} = \frac{ds_{2t}}{ds} = \frac{\kappa_2}{\kappa}\sin\theta, \tag{3.8}$$

$$\left(\frac{ds_t}{\kappa}\right)^2 = \left(\frac{ds_{1t}}{\kappa_1}\right)^2 + \left(\frac{ds_{2t}}{\kappa_2}\right)^2.$$

3.2 The Relationships Between The Developable Ruled Surfaces which Take Parameter Curves and a Curve at The Same Point on a Surface as Directrix Curves

In this subsection, some theorems will be given to present some relationships between the developable ruled surfaces which take parameter curves and a curve at the same point on a surface as directrix curves.

Theorem 1 *Let us given the Darboux vectors of parameter curves* $(v = fxd.)$ *and* $(u = fxd.)$*, which are perpendicular to each other and they are direxrtrix curves of the developable ruled surfaces* $\mathbf{X_1}(s_1, u_1)$ *and* $\mathbf{X_2}(s_2, u_2)$*, respectively, and any curve* $\alpha(s)$*, which is direxrtrix curve of the developable ruled surface* $\mathbf{X}(s, u)$*, that pass through a point of a surface respectively as* $\mathbf{d_1} = \tau_g^{(1)}\mathbf{t_1} + \kappa_g^{(1)}\mathbf{n} + \kappa_n^{(1)}\mathbf{g_1}$*,* $\mathbf{d_2} = \tau_g^{(2)}\mathbf{t_2} + \kappa_g^{(2)}\mathbf{n} + \kappa_n^{(2)}\mathbf{g_2}$ *and* $\mathbf{d} = \tau_g\mathbf{t} + \kappa_n\mathbf{n} + \kappa_g\mathbf{g}$*. Also, let the angle between the unit vectors of the rulings* $\mathbf{l}$ *and* $\mathbf{l_1}$ *be* $(\hat{t, t_1}) = \theta$*. The relationship below can be expressed by using the equations in (16)*

$$\begin{aligned}
\frac{d\mathbf{l_1}}{ds} &= 2\mathbf{t_1}\cos\theta + u_1\left\{\begin{vmatrix} \cos\theta & \sin\theta \\ \tau_g^{(2)} & \kappa_n^{(1)} \end{vmatrix}\mathbf{n} + \begin{vmatrix} \cos\theta & \sin\theta \\ -\kappa_g^{(2)} & \kappa_g^{(1)} \end{vmatrix}\mathbf{t_2}\right\} \\
\frac{d\mathbf{l_2}}{ds} &= 2\mathbf{t_2}\sin\theta + u_1\left\{\begin{vmatrix} \sin\theta & \cos\theta \\ \tau_g^{(1)} & \kappa_n^{(2)} \end{vmatrix}\mathbf{n} + \begin{vmatrix} \cos\theta & \sin\theta \\ -\kappa_g^{(2)} & \kappa_g^{(1)} \end{vmatrix}\mathbf{t_1}\right\}
\end{aligned} \tag{3.9}$$

Proof. Darboux vectors of the directrix curve $\alpha(s) = \mathbf{r}(u(s), v(s))$ of the developable ruled surface $\mathbf{X}(s,u)$ on the surface $\mathbf{r}(u,v)$ are $\mathbf{t} = \mathbf{t_1}\cos\theta + \mathbf{t_2}\sin\theta$, $\mathbf{n} = \mathbf{t_1} \times \mathbf{t_2}$ and $\mathbf{g} = \mathbf{t_1}\sin\theta - \mathbf{t_2}\cos\theta$. Also, for the parameter curves ($v = fxd.$) and ($u = fxd.$), the equations $\mathbf{g_1} = \mathbf{t_1} \times \mathbf{n} = \mathbf{t_1} \times (\mathbf{t_1} \times \mathbf{t_2}) = -\mathbf{t_2}$, $\mathbf{g_2} = \mathbf{t_2} \times \mathbf{n} = \mathbf{t_1}$ and $ds^2 = Edu^2 + Gdv^2$ can be written. Next, the derivatives of the rulings of the developable ruled surfaces in (13) and (14) can be expressed with the equations below.

$$\begin{aligned}
\frac{d\mathbf{l_1}}{ds} &= \frac{\partial l_1}{\partial s_1}.\frac{ds_1}{ds} + \frac{\partial l_1}{\partial s_2}.\frac{ds_2}{ds} = \frac{\partial l_1}{\partial s_1}.\cos\theta + \frac{\partial l_1}{\partial s_2}.\sin\theta \\
&= \left(\frac{d\alpha_1}{ds_1} + u_1\frac{\partial \mathbf{t_1}}{\partial s_1}\right).\cos\theta + \left(\frac{d\alpha_1}{ds_1}.\frac{ds_1}{ds_2} + u_1\frac{\partial \mathbf{t_1}}{\partial s_2}\right).\sin\theta \\
&= \left(\mathbf{t_1} + u_1\frac{\partial t_1}{\partial s_1}\right).\cos\theta + \left(\mathbf{t_1}\cot\theta + u_1\frac{\partial \mathbf{g_2}}{\partial s_2}\right).\sin\theta \\
&= 2\mathbf{t_1}\cos\theta + u_1\left(\frac{\partial \mathbf{t_1}}{\partial s_1}\cos\theta + \frac{\partial \mathbf{g_2}}{\partial s_2}\sin\theta\right),
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
\frac{d\mathbf{l_2}}{ds} &= \frac{\partial l_2}{\partial s_1}.\frac{ds_1}{ds} + \frac{\partial l_2}{\partial s_2}.\frac{ds_2}{ds} = \frac{\partial l_2}{\partial s_1}.\cos\theta + \frac{\partial l_2}{\partial s_2}.\sin\theta \\
&= \left(\frac{d\alpha_2}{ds_2}.\frac{ds_2}{ds_1} + u_2\frac{\partial \mathbf{t_2}}{\partial s_1}\right).\cos\theta + \left(\frac{d\alpha_2}{ds_2} + u_2\frac{\partial \mathbf{t_2}}{\partial s_2}\right).\sin\theta \\
&= \left(\mathbf{t_2}\cot\theta - u_2\frac{\partial g_1}{\partial s_1}\right).\cos\theta + \left(\mathbf{t_2} + u_2\frac{\partial t_2}{\partial s_1}\right).\sin\theta \\
&= 2\mathbf{t_2}\sin\theta + u_2\left(\frac{\partial \mathbf{t_2}}{\partial s_2}\sin\theta - \frac{\partial \mathbf{g_1}}{\partial s_1}\cos\theta\right).
\end{aligned} \tag{3.11}$$

Moreover, known relationships are written as

$$\begin{aligned} \frac{d\mathbf{t_1}}{ds_1} &= \kappa_n^{(1)}\mathbf{n} - \kappa_g^{(1)}\mathbf{g_1}, \quad \frac{d\mathbf{g_1}}{ds_1} = \kappa_g^{(1)}\mathbf{t_1} - \tau_g^{(1)}\mathbf{n} \\ \frac{d\mathbf{t_2}}{ds_2} &= \kappa_n^{(2)}\mathbf{n} - \kappa_g^{(2)}\mathbf{g_2}, \quad \frac{d\mathbf{g_2}}{ds_2} = \kappa_g^{(2)}\mathbf{t_2} - \tau_g^{(2)}\mathbf{n} \end{aligned} \tag{3.12}$$

Hence, if the equations in (24) are substituted in the equations (22) and (23), we obtain the equations in (21). ■

Theorem 2 *Let developable ruled surfaces which take the parameter curves* $(v = fxd.)$ *and* $(u = fxd.)$*, and the curve* $\alpha(s)$ *as the directrix curves at the point* P *on the surface* $\mathbf{r}(u,v)$ *respectively given as* $\mathbf{X_1}(s_1,u_1) = \alpha_1(s_1)+u_1\mathbf{t_1}(s_1)$, $\mathbf{X_2}(s_2,u_2) = \alpha_2(s_2)+u_2\mathbf{t_2}(s_2)$, *and* $\mathbf{X}(s,u) = \alpha(s)+u\mathbf{t}(s)$. *The angle between the rulings* $\mathbf{l}$ *and* $\mathbf{l_1}$ *be* $(\mathbf{l},\hat{\mathbf{l}_1}) = \theta$. *Also, radii of principal curvature of the curve* $\alpha(s)$ *correspond to the directions of unit vectors of the rulings* $\mathbf{l_1}$ *and* $\mathbf{l_2}$ *are* $R_{n1} = \frac{1}{\kappa_{n1}}$ *and* $R_{n2} = \frac{1}{\kappa_{n2}}$, *the radii of normal curvature and geodesic torsion are* $R_n = \frac{1}{\kappa_n}$ *and* $R_{\tau_g} = \frac{1}{\tau_g}$, *respectively. Next, the following relationship can be defined:*

$$\left(\frac{\cos\theta}{R_{n1}}\right)^2 + \left(\frac{\sin\theta}{R_{n2}}\right)^2 = \frac{1}{R_n^2} + \frac{1}{R_{\tau_g}^2}. \tag{3.13}$$

Proof. As mentioned previously, since the parameter curves $\mathbf{r_u}(\mathbf{u},\mathbf{v})$ and $\mathbf{r_v}(\mathbf{u},\mathbf{v})$ are lines of curvature, unit tangent vectors of these curves can be written using the equations in (11) as $\mathbf{t_1} = \frac{r_u}{\sqrt{E}}$ and $\mathbf{t_2} = \frac{r_v}{\sqrt{G}}$. Using the Olind Rodrigue formula, the equations below can be written.

$$\mathbf{n_u} = -\frac{1}{R_{n1}}\mathbf{r_u} = -\frac{1}{R_{n1}}\mathbf{t_1}\sqrt{E}, \tag{3.14}$$

On the other hand, for the unit vector of the ruling $\mathbf{l}$, following equation can be given.

$$\mathbf{t} = \frac{d\alpha}{ds} = \mathbf{r_u}\frac{du}{ds} + \mathbf{r_v}\frac{dv}{ds} = \mathbf{t_1}\sqrt{E}\frac{du}{ds} + \mathbf{t_2}\sqrt{G}\frac{dv}{ds}. \tag{3.15}$$

Since, $\mathbf{t_1}.\mathbf{t} = \cos\theta$ and $\mathbf{t_2}.\mathbf{t} = \cos(\frac{\pi}{2} - \theta) = \sin\theta$, this is true that $\sqrt{E}\frac{du}{ds} = \cos\theta$ and $\sqrt{G}\frac{dv}{ds} = \sin\theta$. Therefore, it is obtained that

$$\frac{d\mathbf{n}}{ds} = -\left(\frac{\cos\theta}{R_{n1}}\mathbf{t_1} + \frac{\sin\theta}{R_{n2}}\mathbf{t_2}\right). \tag{3.16}$$

Moreover, according to derivative formulas of Darboux frame, the equation below can be written.

$$\frac{d\mathbf{n}}{ds} = -\frac{\mathbf{t}}{R_n} + \frac{\mathbf{g}}{R_{\tau_g}}. \tag{3.17}$$

Using the first and second equation in (10), the equation (29) can be rewritten as follows:

$$\frac{d\mathbf{n}}{ds} = \left(-\frac{\cos\theta}{R_n} + \frac{\sin\theta}{R_{\tau_g}}\right)\mathbf{t_1} - \left(\frac{\cos\theta}{R_{\tau_g}} + \frac{\sin\theta}{R_n}\right)\mathbf{t_2}. \tag{3.18}$$

If the equations (28) and (30) are equalized, following equations are obtained as

$$\frac{\cos\theta}{R_{n1}} = \frac{\cos\theta}{R_n} - \frac{\sin\theta}{R_{\tau_g}}, \qquad \frac{\sin\theta}{R_{n2}} = \frac{\cos\theta}{R_{\tau_g}} + \frac{\sin\theta}{R_n} \tag{3.19}$$

Finally, when the square of equations in (31) are taken and added together, we obtain the equation (25). ■

Theorem 3 *Let developable ruled surfaces which take the perpendicular parameter curves ($v = fxd.$) and ($u = fxd.$) with each other as the directrix curves at the point P on the surface $\mathbf{r}(u, v)$ respectively given as $\mathbf{X_1}(s_1, u_1) = \alpha_1(s_1) + u_1\mathbf{t_1}(s_1)$ and $\mathbf{X_2}(s_2, u_2) = \alpha_2(s_2) + u_2\mathbf{t_2}(s_2)$. Additionally, the normal curvatures and the geodesic torsions that correspond to the unit vectors of the rulings $\mathbf{l_1}$ and $\mathbf{l_2}$ are $\kappa_n^{(1)}$, $\kappa_n^{(2)}$, and $\tau_g^{(1)}$, $\tau_g^{(2)}$, respectively. Hence, the following equations can be written for Gaussian and mean curvatures* $K = \kappa_{n1}.\kappa_{n2} = \frac{1}{R_{n1}}.\frac{1}{R_{n2}}$ *and* $H = \frac{\kappa_{n1} + \kappa_{n2}}{2} = \frac{1}{2}\left(\frac{1}{R_{n1}} + \frac{1}{R_{n2}}\right)$

as

$$\frac{1}{R_{n1}R_{n2}} = \kappa_n^{(1)}.\kappa_n^{(2)} + \tau_g^{(1)}.\tau_g^{(2)}$$
$$\left(\frac{1}{R_{n1}} + \frac{1}{R_{n2}}\right)^2 = \left(\kappa_n^{(1)}.\kappa_n^{(2)}\right)^2 + \left(\tau_g^{(1)}.\tau_g^{(2)}\right)^2. \tag{3.20}$$

Proof. We can write the equation (31) according to the unit vectors of the rulings $\mathbf{l_1}$ and $\mathbf{l_2}$ as

$$\frac{\cos\theta}{R_{n1}} = \kappa_n^{(1)}\cos\theta - \tau_g^{(1)}\sin\theta, \qquad \frac{\sin\theta}{R_{n2}} = \tau_g^{(1)}\cos\theta + \kappa_n^{(1)}\sin\theta \tag{3.21}$$

$$\frac{\sin\theta}{R_{n1}} = \kappa_n^{(2)}\sin\theta + \tau_g^{(2)}\cos\theta, \qquad \frac{\cos\theta}{R_{n2}} = \kappa_n^{(2)}\cos\theta - \tau_g^{(2)}\sin\theta \tag{3.22}$$

If we multiply the first equation of (33) with second equation of (34), and the second equation of (33) with first equation of (34), then add them together, it is obtained that:

$$\frac{1}{R_{n1}R_{n2}} = \kappa_n^{(1)}.\kappa_n^{(2)} + \tau_g^{(1)}.\tau_g^{(2)}. \tag{3.23}$$

If the first equation of (33) with second equation of (34) is added together, following equation can be obtained.

$$\cos\theta\left(\frac{1}{R_{n1}} + \frac{1}{R_{n2}}\right) = \cos\theta\left(\kappa_n^{(1)} + \kappa_n^{(2)}\right) - \sin\theta\left(\tau_g^{(1)} + \tau_g^{(2)}\right). \tag{3.24}$$

If the second equation of (33) with first equation of (34) is added together, following equation can be obtained.

$$\sin\theta\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \sin\theta\left(\kappa_n^{(1)} + \kappa_n^{(2)}\right) + \cos\theta\left(\tau_g^{(1)} + \tau_g^{(2)}\right). \tag{3.25}$$

Finally, if the squares of equations (36) and (37) are taken and added together, the second equation in (32) is obtained. ■

4 Conclusions

Developable ruled surfaces and lines of curvature are very important for CAGD and engineering. Many researchers focused on these topics for a long time. Therefore, we studied on surfaces based on developable ruled surfaces. We took parameter curves of a surface as lines of curvature and a curve at the same point on a surface. Next, we analyzed a surface using three developable surfaces that take these parameter curves and a curve at the same point as directrix curves. For this purpose, we presented the tangent indicatrices of the directrix curves of the developable ruled surfaces according to parameter curves and a curve related with their arc length parameters at the same point on a surface. We gave the relationship between the rulings of the developable ruled surfaces that take parameter curves as directrix curves at the same point according to their geodesic curvatures and geodesic torsions, and angle between the unit vectors of the rulings of the developable ruled surfaces that take a curve and a parameter curve ($v = fxd.$) as directrix curves at the same point in Theorem 3.1.
In Theorem 3.2, the relationship between the radii of principal curvature correspond to the directions of unit vectors of the rulings of the developable ruled surfaces that take the parameter curves as directrix curves and the radii of normal curvature and geodesic torsion of the developable ruled surface that takes a curve as directrix curve at the same point is expressed. Also, we gave the relationship between Gaussian and mean curvatures of a directrix curve of a developable ruled surface and the normal curvatures, the geodesic torsions of the parameter curves which are taken as directrix curves at the same point on a surface. Hence, each regular surface can be examined based on developable ruled surfaces.

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TOPOLOGICAL STRUCTURES IN PARTICLES AND SPIN HYPOTHESIS

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Abstract

First, topology and its applications are briefly described. Next, we propose the topological structures of particles which determine different spins, and two trefoil knots are possibly similar to two spins s=±1/2 of three quarks (nucleons p and n, etc). Further, this is extended to general fermions and bosons. It may contribute to resolve the spin crisis. Third, based on the topological structures of particles, we research possible violation of Pauli Exclusion Principle (PEP), and propose some predictions. Finally, we discuss other effects of topological structure, and some applications.

Keywords: topology; particle; spin; fermion; boson; Pauli Exclusion Principle; magnetic momentum.

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I. Introduction

Topology is the mathematical study on properties of space preserved under continuous deformations including stretching and bending, but not tearing or gluing. It includes connectedness, continuity and boundary. Topology is widely applied to economics, engineering, DNA, heart stimulation, population modeling, cosmology, and computer graphics, degree theory, dynamical systems and chaos, graph theory, metric spaces, connectedness, compactness, etc [1,2].

In 1964, Wheeler, et al., edited *Relativity, Groups, and Topology* [3]. Topological physics includes many regions in physics. Geometric topology includes knot theory. Kauffman investigated specially knots and physics [4]. Topology is also applied to particle physics. AB (Aharonov-Bohm) effect in quantum mechanics is namely a topological effect. Polyakov discussed quark confinement and topology of gauge theories [5]. In 1978 Witten, et al., used the corrected supersymmetry algebra with solitons, which includes the topological quantum numbers as central charges [6]. Osborn obtained the central charges in the supersymmetry algebra of the N=4 supersymmetry gauge theory, and the spins of the topological monopole states are identical to those of the massive elementary particles for spontaneous symmetry breaking [7]. Alvarez discussed theory of strings with boundaries, and its fluctuations, topology and quantum geometry [8]. In 1991 A.P. Balachandran researched classical topology and quantum states.

Aharonov and Casher shown neutral particles with a magnetic moment exhibit the AB topological effect in the nonrelativistic limit [9]. Witten discussed topological tools in 10-dimensional physics [10]. Eguchi, et al., introduced N=2 superconformal models as topological field theories [11]. Birmingham, et al., researched systematically topological field theory, in which includes evaluation of the partition function, evaluation of the observables and knot, super-symmetric quantum mechanics, topological sigma models, and connections with conformal field, etc [12]. Protheroe, et al., searched the propagation of ultra high energy protons and gamma rays over cosmological distances and implications for topological defect models [13]. In 1998 D.J. Thouless published book *Topological Quantum Numbers in Nonrelativistic Physics*. In this paper, we propose that the topological structures of particles determine their spins, and discuss its possible applications.

2. Topological Structures of Particles

Weinberg used topology in extended field configuration of quantum theory [14]. He pointed out: It often happens that the space of all possible field configurations may be given a non-trivial topology by the condition that some functional S of the various fields is finite. In particular, extended configurations with a different topology from the usual spatial uniform vacuum fields cannot

spread out to become spatially uniform. Some examples include [15,16] domain boundaries, Skyrmions, instantons, monopoles, etc. It may be produced in symmetry-breaking transitions in the early universe, in which they are known as homotopy theory [17] and cosmic strings [18].

Ooguri and Vafa searched knot invariants and topological strings [19]. Hill researched topological physics of little Higgs bosons [20]. In 2020 Zheng, et al., researched three-dimensional higher-order topological acoustic system with multidimensional topological states [21]. Yan studied unconventional topological insulators from extended topological band degeneracies [22]. Dahan, et al., searched dynamically induced topology and quantum monodromies in a proximity quenched gapless wire [23]. Zhang, et al., discussed highly tunable topological system based on PbTe-SnTe binary alloy [24]. We discussed the topological physics and the general topological sciences [25]. String and its developments, some new questions on superstring and the emergence string are researched [25], and the mass-life formulas of particles may be derived from the emergence string [26].

An inextricable knot in topology corresponds to that a quark cannot be decomposed. This includes string, brane and bag, etc., of particles. These models are all some topological structures [27], and may be the topological manifolds of small extra dimensions. Bosons as quark pair can only be open strings, and fermions as three quarks can form closed strings.

3. Spins Determined by Topological Structures

It is known that many particles and resonances are unified by Regge trajectories [28], and mesons and baryons are unified by quarks. In quark model various different particles can be classified as difference of quarks. But, so far any free quark is not observed. The baryons of three quarks may be described by the Borromean rings with three loops in topology (Fig.1) [29-33], here united they stand, divided they fall [34]. Further, quarks are developed to preons or subquarks, even prepreons.

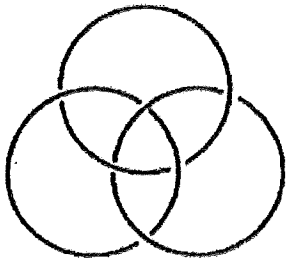


Fig. 1. Borromean rings

The orbital angular momentum L is not conserved, and the total angular momentum J=L+S includes the spin S. But, the spin of particle cannot be real

rotations of its constituent. Any particles with spins, in particular, electron, shows that they cannot be point-particles, which cannot and have not rotations. Therefore, spin predicts these particles have necessarily structures. It is a puzzle of spin. The spins are affected by magnetic field, and were shown in Stern-Gerlach experiment, anomalous Zeeman effect and so on

In knot theory of topology, a trefoil knot is the simplest non-trivial knot. We develop the Borromean rings model of baryons with three quarks, and propose that a trefoil knot of left hand and a trefoil knot of right hand (Fig.2) is possibly similar to two spins s=1/2 and -1/2 of three quarks (nucleons p and n, etc). Basic mesons of two quark form a line, and s=0. This model is first applied to proton and neutron, then may be extended to other particles.

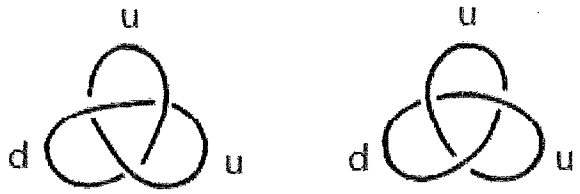
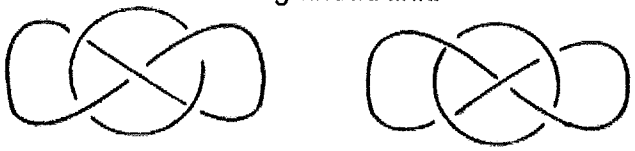


Fig.2 A trefoil knot of right hand and a trefoil knot of left hand

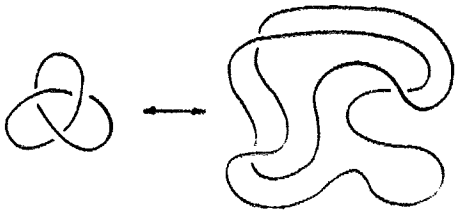
In molecular structures the trefoil knot is the topological chiral molecules with D3 symmetry, and forms a non-planar C-C conjugate and twisted inner spiral structure, which have weak bond at the intersection and distort the molecular orbits. It is also similar to the left and right isomers of biomolecules, and to the cholesteric in soft matter, etc.

Moreover, topology has 8-shaped knots:

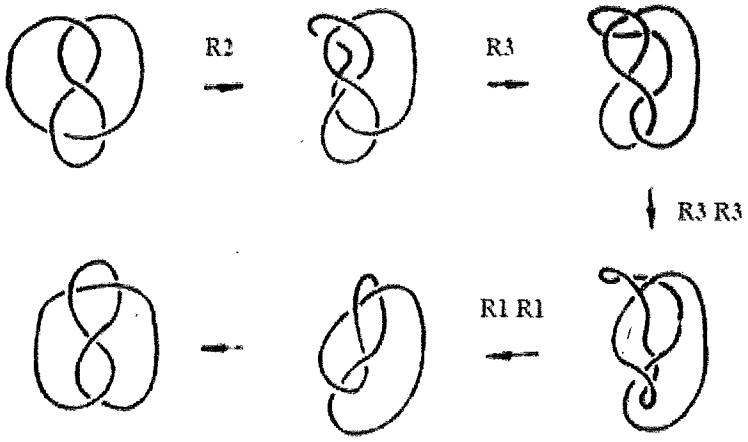
Anti-Wighthead link and Positive-Wighthead link:



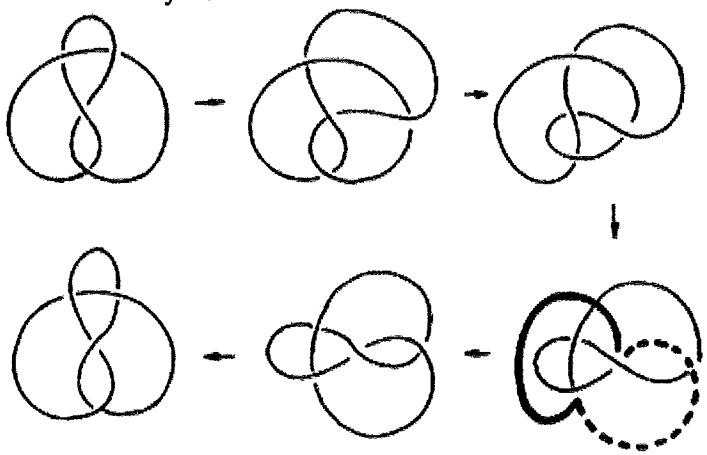
"Plane deformations" are:



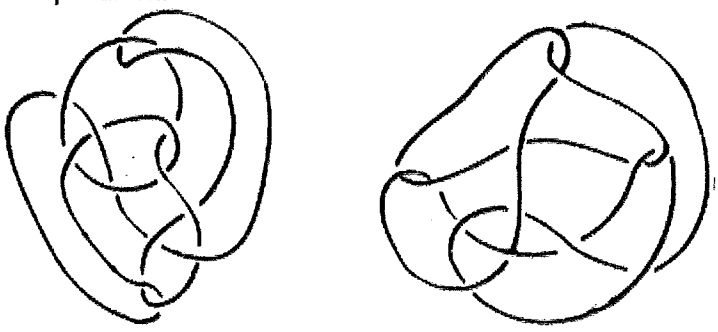
The 8-shaped knot is equivalent to its mirror image:



General deformation may be:



In 1985, it has been proved that the following two 13-point nodal projection diagrams are equivalent.



Assume that spin s corresponds to the Betti number

$$B = 1 + L - V. \qquad (1)$$

Or we may introduce a new topological quantum number of particles to connect quark number and spin number, etc. For example, let s=B/2, here vertex V is quark number, edge L is gluon number and ways of interactions. The basic baryons V=3, L=3, s=1/2; mesons V=2, L=1, s=0. Probably, it corresponds to some structures of Feynman diagram. The vertex V adds quark pair, s is invariant or s+1. The fermion s=1/2 has left and right mirror symmetry, and the boson s=0 has no left and right mirror symmetry.

So far, various experiments have shown some spin crisis. Since the mid-1980s the spins of particles should not be attributed entirely to the spin of its component quarks. It has been confirmed before July 1999 that only 30% of proton spin is produced by its quark spins; the remaining 70% comes from the spin of the gluon and the orbital angular momentum of all quarks and gluons in the nucleons. The anomalous spin is mainly considered as the contribution of see quarks, the contribution of gluon, valence quark and the angular momentum of gluon. The theory is axial vector current theory [35]. This is an anomalous theory. The correlation coefficients are introduced for the spin distribution function [36].

In 1993 SLAC held conference on *Spin Structure in High Energy Process*. M.E. Peskin researched why is there spin? More precisely, why do the basic pointlike constituents of Nature carry intrinsic nonzero quanta of angular momentum? From the general Poincare group he discussed spin until based on the rotation. C.Y. Prescott discussed the spin physics, which plays an important role in high energy experiments and theories.

Pasechnik, et al., researched the nonperturbative and spin effects in the central exclusive production of the tensor meson $\chi_c(2^+)$ for proton-antiproton collisions [37]. Takahashi, et al., construct 2×2 cross-correlators from flavor SU(3) octet and single baryon operators, and discussed the low-lying Λ baryons with spin 1/2 in two-flavor lattice QCD [38]. Wang, et al., calculated the nonleptonic weak processes in spin-one color superconducting quark matter [39].

A theoretical problem is in quantum chromodynamics (QCD), which has nonlinear, strong interactions (not perturbation) and self-interactions between gluons, which can continuously create and annihilate virtual particles. When the mass of the same particle and its resonance increases, the spin is also changing, but both relations are very complex, so it is difficult to find a fixed law. Mass is probably increased by two ways: (1) An equal-difference series, i.e., increasing mass of a quark pair, which is about 100MeV. (2) Add An^2 related to oscillation and rotation, etc.

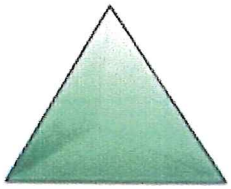
4. Some Prediction, PEP and Its Possible Violation

The Pauli Exclusion Principle (PEP) is a very important natural rule, whose base is spins of fermions. PEP is widely applied and may explain from the periodic table of elements to neutron stars, etc. But, some scientists queried the universality of PEP some times. Santilli first proposed the rest of PEP, in particular, for strong interactions, and the structure of hadrons, nuclei and stars [40-42]. Further, Santilli researched inapplicable of PEP in numerous experiments [43]. Based on some experiments and theories of particles at high energy, we investigated violation of PEP at high energy begin from 1984 [44-51], and researched some possible tests on violation of PEP [45,50,51]. Recently, we discussed another possibility on violation of PEP (VIP) at ultra-low energy, which is similar to superconductivity, superfluids and Bose-Einstein condensation (BEC), etc., and proposed a possible mechanism of VIP: Cooper pairs extend to general fermion pairs from lepton pairs to nucleon pairs (n-n more easy than p-p), so they transform to bosons with VIP. From this we predicted some characters of this like-boson, as in nuclei and atoms, etc., and researched some possible VIP in mathematics and physics. Based on the extensive quantum theory, its PEP may be violated [52].

It is known that at very low temperature two fermions can constitute a boson like the Cooper pair, and perform BEC. In 1995, the condensation numbers of ^{87}Rb and ^{7}Li atoms may be high as 10^5 under this extreme condition. In 1999 DeMarco and Jin cool the potassium atom gas with fermion characteristics to more than one billionth of an absolute zero, and the potassium atom pairing can realize quantum degenerate of fermi atomic gas [53]. It has variety of quantum effects with different bose atomic gases, as Fermi pressure, Pauli blocking and superfluid, in which ^{3}He is fermion, and cannot form BEC. But, ^{3}He may pair form superfluid. Further, any fermions can become bosons in pairs, and form superconductivity and superfluid. Some interactions derive to form similar Cooper pairs of fermi atoms, and change resonance interaction to achieve the BEC phase transition from the BCS superfluid [54-56]. Fermions pairing at ultra-low temperature can violate PEP. In this case PEP has not played a role in the ultracold structure, and violation of PEP may be tested.

Conversely, in 1960 Girardeau proposed the gas model under the Bose limit of a hard core, i.e., the Tonks-Girardeau gas, where bosons are limited to one-dimensional space, and have stronger repulsion, which is similar PEP. In 2004 Paredes, et al., confirmed this result by cold atomic experiments [57]. This is bosons like fermions, so both show symmetry and unification.

From this hypothesis of spin we can predict: 1. Spins as the plane-space topological structures must cause the change of angular momentum. For example, a structure of boson with four quarks is possibly a regular tetrahedron and s=1.



2. Topological structures are different or change, so spins are also different or change.

3. The spins of particles with many quarks are complexity.

4. If it is developed into 3-dimensional space, topology and its application will be more complex. The ground state of two quarks determines a straight line, and three quarks determine a plane. Four or five quarks-gluons can form space, and multiple quarks-gluons produce complex topological structure.

5. Spin can unify fermions and bosons, which includes that the spin term of equations can be neglected [46]. So we proposed violation of PEP at high energy [44-51] and at ultra-low energy and possible mechanism [52].

6. Leptons with spins should have structures.

7. The trefoil knot is the twisted inner spiral structure, which is possibly related to the magnetic moments of particles.

8. It is probably related to the difference of electromagnetic masses of particles.

Coleman proved the 1+1 dimensional Sin-Gordon equation is equivalent to the mass Thirring (MT) equation (corresponding Dirac equation) under certain conditions [58], and its soliton is always equal to the basic fermions of the Thirring model. In (1+1) dimensional space-time fermion and boson are connected each other. In this case spin may actually be neglected. Low-dimensional two particles are correlation, and high-dimensions should also be associated, as string model (D=26 dimensions). For interactions, especially when the interaction is strong and for many-bodies, spin may be arbitrary, which corresponds to anyon and violation of PEP. These possibly correspond to many cluster effects with spins.

Most fundamentally, experimental facts have revealed that conventional spin loss, and proton spin becomes zero. It is related to at high energy spin can be neglected [46]. QCD believe that the proton spin effect at high energy should be equivalent to zero. Fermion and boson are unified quantum statistics [44]. This seems to have been confirmed by Fermilab and CERN experiments, in which proton energy about 200~400GeV. It is the same with the basis of unified statistics and violation of PEP predicted by the author, in which the spin effect can be neglected, and had the same threshold [44,46]. QCD is strong interaction, and suitable for high energy. At this case, the spin is probably zero, and unified statistics.

Violation of PEP is associated with unified statistics [44] and spin of particles. Spins may have theoretical fraction and irrational numbers, and experimental spins. The spin of particles at low energy (such as proton) is determined by valence quarks, but most of spin are determined by gluons at higher energy [59]. This is not conducive to the quark model. Probably, higher energy may be similar to mesons as zero (such as QCD), or fractional spin, and will tend to unify.

In a word, spin deviations and variability are all related to the possible violation of PEP and significant changes in quantum theory. It has hinted that quark model and QCD should be developed.

The most fundamental basis for violation of PEP is the experimental facts. At present, experiments and theories estimate $\beta^2 \leq 10^{-30} \sim 10^{-6}$. If the proton spin is zero at high energy, statistics will be unified; if it is not zero, we must develop QCD. This is linked to collisions. Generally, collisions tend to be constant when the energy reaches a certain range, i.e. Pomeranchuk theorem, and there is scaling invariance, so there is invariant fractal dimension. In nonlinear theory, solitons collisions and the shapes remain unchanged, which show that they are compatible each other. This can be related to violation of PEP, and correspond to the nonlinear theory [47].

We believe PEP will be violated under certain conditions [44-52]. Except some tests propose before, combining above discussion we can predict that two dimensional electron gas and two dimensional isospin ferromagnet correspond to 2+1 dimensional nonlinear σ model, so all the systems agreed with the nonlinear σ model, such as two dimensional electron gas and two dimensional isospin ferromagnet, etc., should have fractal spins according to the mathematical conclusion, and obey the singular statistics, and violate PEP.

It is known that two electrons in the helium atom are spins in opposite directions. The influence of an external magnetic field forces the same spin orientation. But, the theory predicts that one of electrons must jump to a higher energy level. Quantum dot can be tested with two Tesla magnetic field (the real atom must have 400000 Tesla magnetic field), and high accuracy detects quantum chaos and quantum Hall effect. Ashoori, et al., applied single electron capacitance spectroscopy, and discussed N-electron ground state energy of a quantum dot in a magnetic field, and drew the ground state energy of from 0 to 50 electrons determined by magnetic field [60]. This leads to quasi-two-dimensional character, and corresponds to fractional quantum Hall effects, anyon and violation of PEP. Moreover, by laser tweezers single atom may test violation of PEP.

Further, we can combine several investigations in total physics, such as violation of PEP, singular statistics, fractional spin, anyon, nonlinear theory, supersymmetry, chaos, etc. These seem to be able to be unified by nonlinear

theory. Various statistical partition functions compare, which are related to spins with fraction and irrational number. Even, spin may be complex number. All models of fractional statistics can violate PEP, and fractional statistics and spin may correspond to fractional dimensions. Conversely, the author's nonlinear theory [46,61] should lead to fractional spin, and its mathematical representation is the fractal dimension matrix [62,63]. By unified statistics and Gamma distribution it can be related to infinite statistics and so on.

We propose that spin of hadrons is a topological concept. It may correspond to different structures, angular momentums and magnetic momentums, which may combine $\mu_n / \mu_p = -2/3$ [64], and my magnetic moment formulas [65].

Koures, et al., discussed spontaneous parity violation in a supersymmetric nonlinear σ - model in 2+1dimensions [66]. Such violation of PEP corresponds to the nonlinear σ model, and derives parity violation, so two violations seem to be related. Further it may also be related to violation of CP, etc. PEP has been linked to conservation of PCT.

5. Other Effects of Topological Structures and Some Possible Applications

For spin the main experimental results include: A.D. Krisch, et al. found in May 1984 by 10GeV and July by 14GeV, when the spin of the incident proton accelerates, large angle scattering appear, i.e., the chance of violent head collision is three times that of slow proton spin. According to usual theory, there is no such selection process. Based on QCD, the large angle deviation should be the direct interaction between quarks-quarks in protons. This should not be affected by the total spin direction of proton. According to the assumption in this paper, the spin change represents the topological structure change of particles, such this is entirely possible.

Anselmino, et al., searched single spin asymmetries in lepton-polarized proton $lp^{\uparrow} \rightarrow hX$ and $lp^{\uparrow} \rightarrow$jet$+X$ processes [67]. Zhou, et al., searched the relation between spin and quark distribution in large transverse momentum, and calculated the terms of quark-gluon correlation functions and the angular distribution of polarized Drell-Yan processes [68].

The polarized proton experiments at 12GeV shown when the collision energy increases, the spin effect does not decrease and seems to be more important. 1) The reaction cross section between the spin parallel protons under large momentum transfer is much larger than that of inverse parallel cross section, scattering angle is more than 60°, when transverse momentum $P_{\perp} >$ 3.3GeV the ratio suddenly rises, and to 4 for 90°. 2) For total cross section between longitudinal polarized proton beam and the target proton, the parallel collision cross section has about 50% larger than the flat antiparallel

section at momentum about 1.5GeV. We can speculate that 1.5GeV is probably a threshold, later tend to be consistent. 3) The neutron-proton spins are opposite, the interaction cross section is larger, which are just opposite to the proton-proton collision. This seems to be the same proton-proton repulsed each other, so the cross section is larger.

Proton-proton polarization scattering indicates that spins have more complex essential properties. Possibly, 1) the particle or even spin consists of more basic components. 2) The spin direction is related to the interaction, which is similar to the conductor current absorbs in the same direction and repels in the reverse for electromagnetic interaction. We may suppose that new interaction exists for the particles spin, which is attached to the pp interaction. As the energy increases, or asymptotically free, the interaction of pp or subparticles in proton decreases, or various interactions tend to be unified, and the spin force increases. So the spin effect is obvious.

First, we should summarize and analyze the laws of particle spins and their collision, and how it changes with energy. Then, combined with various properties of particles at medium and high energy, such as collision of different particles tend to be unified, new fractional statistics, possible breaking of some principles, existence of nonlinear effects, etc., and their corresponding theories, unification of equations, nonlinear quantum mechanics, nonlinear σ model, and explore the possible mechanism of particle spins and their variations with energy. A new polarization phenomenon may indicate that P is not conserved in some cases. From this searches the range of proton spin, whether it is related to the change to Δ resonances.

In the nonlinear σ model the total angular momentum is:

$$\text{J=M(orbital angular momentum)+}\frac{\theta}{2\pi}Q^2\text{ (corresponding fractal spin). (2)}$$

Here θ is any function. This can be fermions or bosons, or more general particles with arbitrary spin. $\frac{\theta}{2\pi}Q^2$ is an uniform representation of the spin of these particles. When θ is variable, they can transform each other.

Equations at high energy may be unified [46], so spins can be: 1. The spin term can be neglected. 2. The mass term can be neglected, so spin can be 1/2 of the neutrino and 1 of the photon, etc. 3. Spin can be arbitrary. 4. Spin is symmetry for supersymmetric theory. This is related to spin at high energy can be neglected. If the spin and mass of particles can be neglected, the statistics will be unified [46].

Therefore, (1) It is probably related to spin at low energy, while the effective spin at high energy is zero. (2) Spin s=0 at the particle level, but there is still spin effect s≠0 at the deeper level of quark-parton, subquark, etc. The particle is at high energy, the subparticles may still be at low energy, only their

number increases, or their energy is not high enough to reach the energy scale that the spin effect can be neglected. The high energy polarization makes just the spin direction of the subparticles is the same. In this way, spin of this particle should also become larger and has several effects. The transition from s=0 to s≠0 corresponds to open the next level, and corresponds to the phase transition point. This is that particles, subparticles are excited at high energy. It may explain simultaneously unification and new spin effects at high energy.

If proton spin is different at different energies, it seems to be only: 1. Spin is not fixed, and can change and be more spin quantum states. 2. Components in particles (such as quarks) can be more regular and orderly, which corresponds to the phase transition. Such as p=uud, spin can be 1/2, 3/2; if there are 2n+1 fermions in the proton, the spin can have (2n+1)/2 quantum states. This assumes that quark spins are still in two directions. If quarks and so on can also have more spin directions, it will be more complex.

Based on the energy and angular momentum, etc., proton s=1/2 composed of quarks is the most favorable, and the energy is the lowest. Phase transition at high energy can form protons with higher energy and larger spin. This is the liberation of spin degrees of freedom. s=±h/2→±(2n+1)h/2. If ratio $d\sigma/dt$ equals ratio of spin-energy, or ratio of spins, $d\sigma/dt$ will be quantized, and jump. Conversely, $d\sigma/dt$ and spin will be not quantized.

First, it should be clear whether proton spin will change? How to change? Open different numbers of quarks, subquarks can have different spins. Second, large spin shows more quarks or subquarks in proton, so the chance of head-to-head collision is larger. The two aspects may be classified as subparticles in proton, such as shown subquarks, of course the number is larger.

At the same time, spin and interaction are related to collision. Their properties and energy are between s→0 and fixed s for different energies. If the energy is less than 1GeV, $p\bar{p}$ scattering is significantly associated with the spin axis of collision particle being parallel or antiparallel. Generally, collisions tend to be constant when the energy reaches a certain range, and have scale invariance and fixed fractal dimension. Possibly 1. Spin at high energy s→0, and corresponds to σ →constant. This is unification of all fermions and bosons for the whole particle. 2. For very high energy s≠0, and corresponds to σ rises. This is due to the existence of subparticle (quark etc.) structures inside particles, whose effects have shown similar EMC effects. Spin is probably the smallest ground state zero angular momentum and its excited state, which is a quantum character, but is not necessarily real rotation.

Particles with different spins have different interactions, i.e., the scattering cross section and the angular distribution of decay are related to particle spins.

In this experiment, polarized particle beams are used. They often tend to be the same at high energy.

It corresponds possibly to that Λ, Σ^0 (d-u-s) with the same quarks have different rotations. The neutrinos have only left or right rotation. $K_L^0, K_S^0 (d\bar{s})$ have different lifetime, and PC and T non-conservation. $D^0 (c\bar{u}), B^0 (d\bar{b})$ are symmetry, so they should have corresponding character.

Mesons with two quarks form a line, which is similar to magnetic poles. Fermions with three quarks, it possibly linked to magnetic moments. One-dimensional p(u-d-u) and n(d-u-d), three quarks in plane correspond to triangles, but the two angles of u-u are mutually exclusive, the charges at both ends of proton are 2/3, and the repulsion is larger; and the charges at both ends of neutron are -1/3, and the repulsion is smaller.

The both angles are different, and the magnetic moments are also different. This is probably related to the electromagnetic mass difference. Σ^+ (u-s-u), but Σ^- (d-s-d) is mutually exclusive, and mass is the biggest. $\Delta^{++}(uuu), \Delta^-(ddd)$ and $\Omega^-(sss)$ mutually exclusive. $\Omega^-(sss)$ mass is the biggest. Multiple quarks, such as $u\bar{d}(\pi^+) + u\bar{u}$ and $udu(p) + u\bar{u}$ may become two bodies.

Klein-Gordon (KG) equation is:

$$[c^2\hbar^2 \frac{\partial^2}{\partial x_\alpha^2} - \hbar^2 \frac{\partial^2}{\partial t^2}]\psi - m^2c^4\psi = 0. \qquad (3)$$

$$[\frac{\partial^2}{\partial x_\mu^2} - \frac{(mc)^2}{\hbar^2}]\psi = 0. \qquad (4)$$

It describes bosons with spins s=0, ... n, which include 2 or 2n quarks.

Dirac equation is:

$$\gamma_\mu \partial_\mu \psi + \frac{mc}{\hbar}\psi = 0. \qquad (5)$$

It describes fermions with spins s=1/2, ... n/2, which include 3 or 3+2n quarks. The spins at a certain degree can unify fermion and boson s=n/2.

In spherical coordinates,

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+[x^2-(j+\frac{1}{2})^2]y=0. \qquad (6)$$

It is j+(1/2) order Bessel equation. Its solution is $y=Ax^{j+(1/2)}+Bx^{-j-(1/2)}$.

In column coordinates,

$$x^2\frac{d^2R}{dx^2}+x\frac{dR}{dx}+[x^2-m^2]R=0. \qquad (7)$$

R(x) is m order Bessel equation. Eqs. (6) and (7) correspond to fermions and bosons, respectively [65]. Both possess the spherical and column symmetries.

In A.D. Krisch, et al., polarization pp scattering experiments with the energy range of 4~13GeV, from 4 to 8GeV energy increases with the incident proton energy, the section drops sharply and straightly, the spin direction is parallel and anti-parallel, and slow down above 8GeV. When two protons spin parallel, the section is bigger, at 13GeV parallel/antiparallel ratio is 4, and QCD ratio should be 2, here is 90° scattering angle. It can assume that proton component is not exactly quarks, or three quarks in pp collision contact each other, and are not the asymptotic freedom. Anyway, the high energy (≥8GeV) seems to be not known quark. The polarized proton beam experiments at 18.5 and 22GeV show that the spin parallel/inverse parallel cross-section ratio oscillates up and down at certain scattering angle, or certain incident proton beam energy. N. Tyurin theory predicted such results. Experiments also show that the cross-section ratio varies with energy, but for certain incident proton energy the maximum all lie on 90°. It is predicted by H. Lipkin model. In 1983 the number of protons scattered to the left and to the right in the elastic scattering of polarized targets by unpolarized proton beams is equal, which agrees with QCD. But, the number of protons scattered to the left in the large angle scattering at 28GeV is two thirds higher than that in the right scattering, which do not agree with QCD. This indicates that the theory of high energy should be different!

The known collision sections, especially for quantum electrodynamics (QED), tend to be unified at very high energy, which correspond to unified statistics [44,46]. Collisions of QCD and various interactions should also have unified form at very high energy, and correspond to unified statistics and GUT, etc., and the collisions of various interactions should be unified, especially in the unified scale. The quantum field with short distance is namely one at very high energy. At very high energy, it will be: 1. degeneration and unification, 2. statistics and random fluctuations. 3. it is possibly chaos. Results are the same for short distances.

In the experiments, various collisions at high energy, regardless of interaction, tend to be constant and rise. The mass term at high energy can be neglected, and the equations are unified to:

$$\gamma_\mu(\beta_\mu)\partial_\mu\psi = J. \qquad (8)$$

By the soliton method the equation (8) may further be simplified to $\psi' = J$. Here two simple cases are: 1. When J=0 (there is no interaction, asymptotic freedom), ψ=C, and section $\sigma = C_0$ tends to be constant. It corresponds to Pomeranchuk theorem [60]. 2. For very high energy interactions are unified, and interaction constants tend to be the same, J=C, so ψ=CE, and the cross-sections $\sigma = C_0 E^2$ rise with energy.

Generally, σ model can be both nonlinear and supersymmetric. The nonlinear σ model of pure mathematics leads to fractional spin, and singular statistics leads to unified statistics, such as Gamma distribution [44]. 3+1 dimension and 2+1 dimensional nonlinear σ models are different. The 2+ε+1 dimensional nonlinear σ model should be discussed.

For three quarks the left and right spin is related to charge. And it can also be related to the origin of magnetism and monopole.

But, it cannot explain spins of electron e and neutrino ν_e as point-particles, whose rotations are more impossible. Such spins of leptons should be extended as three subquarks components. Probably, e-ν_e are similar to p-n, Let similar subquarks u'=(-2/3)e, d'=(1/3)e, so $e = u'u'd'$, $\nu_e = u'd'd'$. In 1963 Dirac proposed that electron has structure, so μ, τ are its excited states. Further, two neutrinos ν_μ, ν_τ and (c,s) (b,t) are all these excited states of ν_e and (u,d). Three generations of lepton-quark form a homology group (and curves).

The topological solitons [70] exist in one dimensional sine-Gordon equation [71-73], two dimensional Nielson-Oleson vortex field [74] in relativistic quantum field theories, three dimensional 't Hooft-Polyakov monopole solution [75,76], etc. The sine-Gordon equation is:

$$\phi_{xx} - \phi_{tt} = \frac{1}{2}\pi\lambda^2 \sin(2\pi\phi). \qquad (9)$$

Its soliton solution is:

$$\phi = \frac{2}{\pi} tg^{-1}[\exp(\pm\pi\lambda \frac{x - ut}{\sqrt{1-u^2}})]. \qquad (10)$$

6. Discussions

The global topology exists in the field manifold. Topology is related to nonlinearity and differential equation, etc. In topology there is fixed point. Lefschetz fixed point theorem is in n dimensional complex manifold. Its special

case is that Brouwer fixed point theorem which is true spherical in n dimensions. Fixed point is f(x)=x, so it is related to chaos.

Three fermions p, e, ν_e and photon γ are only four stable particles. They are four basic fixed points with four interactions. No matter how excited, collided, interacted, as long as particles are bounded, and are not broken, they must have a fixed point in string, brane, bag and M models.

The Pomeranchuk theorem assume that all quark-quark and quark-antiquark scattering amplitudes are asymptotically equal, which is valid at high energy [64]. The structures and collisions in particles should show some topology. It can even include three generations of quark-leptons, whose topological invariance corresponds to symmetry of three generations, etc.

This hypothesis may be applied to spins and polarizations in atoms. The relation between it and quantum number I, S, etc., should be researched, which possibly determine the decay modes.

The knots and kinks, etc., in particles should be studied. These could correspond to spinor, twistor and Regge poles, Veneziano model, etc. Various scaling and R-(1/R) duality etc., are all the topological invariance.

In a word, it will be wide and important that topology is applied in particle physics.

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